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Denoising Manifold-Valued Data by Relaxed Regularizations

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Funded by:
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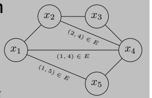
Problem: Variational Inverse Problem

- **Denoising** signals on graph $G = (V, E)$



- Manifold-valued data $x_n \in \mathbb{M} \subset \mathbb{R}^{d_\mathbb{M}}$
- Variational model with denoiser $\mathcal{D}$
  a) Tikhonov regularization
  b) Total Variation (TV)

$$\mathcal{D}(\boldsymbol{x}_n, \boldsymbol{x}_m) \in \{1/2\|\boldsymbol{x}_n - \boldsymbol{x}_m\|_2^2, \|\boldsymbol{x}_n - \boldsymbol{x}_m\|_1\}$$

- Original problem:

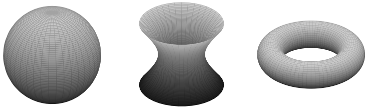
$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{M}^N} \tfrac{1}{2} \sum_{n \in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \lambda \sum_{(n,m) \in E} \mathcal{D}(\boldsymbol{x}_n, \boldsymbol{x}_m)$$

Second Step: Convex Relaxation

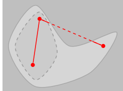
- **Neglect** $\operatorname{rk}(\boldsymbol{Q}_{(n,m)}), \operatorname{rk}(\boldsymbol{V}_n)$
- Relaxed **convex** problem with
  a) Linear objective (Tikhonov)

$$\mathcal{D}(\boldsymbol{x}_n, \boldsymbol{x}_m) = \tfrac{1}{2}(\boldsymbol{x}_n + \boldsymbol{x}_m) - \boldsymbol{\ell}_{(n,m)}$$

  b) Linear fidelity term (TV)



Main Difficulty: Nonconvexity



- Constraints $x_n \in \mathbb{M}$
- Lead to **nonconvex optimization problem**
- Hard to solve globally
- Standard convex methods cannot be applied directly

Third Step: Simple Computations

- ADMM, due to (partial) linear

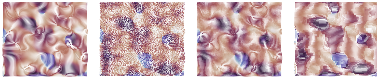
$\rightarrow$ critical points or $\operatorname{prox}_{\mathrm{TV}}$ dim.-wise

$+$ projection on convex cone of PSD matrices $\rightarrow$ truncated SVD

General Framework, Many Manifolds

- spheres $\mathbb{S}_{d-1}$
- torus $\mathbb{T}_d$
- rotation $\mathrm{SO}(d)$
- frames $\mathrm{SE}(d)$
- hyperboloid $\mathbb{H}_d$
- Stiefel $\mathbb{V}_d(k)$
- binary $\mathbb{S}_0^d$ + theoretical "tightness"

"tightness" can be observed in praxis



First Step: (partial) Linearization

- Introduce auxiliary variables

$\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle, \boldsymbol{v}_n = \|\boldsymbol{x}_n\|^2$ and from def. of $\mathbb{M}$

- Encode constraints by **matrix representation** $\boldsymbol{Q}_{(n,m)}, \boldsymbol{V}_n \succeq 0$

$+$ minimal rank property $\boldsymbol{Q}_{(n,m)}, \boldsymbol{V}_n$

- Equivalent problem:

$$\operatorname*{arg\,min}_{\boldsymbol{x},\boldsymbol{v},\boldsymbol{\ell},\bullet} \sum_{n \in V} \tfrac{1}{2} \boldsymbol{v}_n - \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle + \lambda \sum_{(n,m) \in E} \mathcal{D}(\boldsymbol{x}_n, \boldsymbol{x}_m)$$