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Generalization of Normalized Radon Cumulative Distribution Transforms in the Light as Feature Extractors for Image Classification

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Motivation: Objects as Prob. Measures

- Classification with **many classes** and **few samples**
- Assumption: (approx.) **affine** / structured class families
- Need: a) **invariance** to affine transformations
  b) **stability** to deformations
- Application: historic watermarks, automated pattern recognition

Normalization: NR-CDT and Sorting

- For $\mu \in \mathcal{P}^{+}_{\phi,\mathrm{c}}(\mathbb{X}) \subset \mathcal{P}_{\mathrm{c}}(\mathbb{X}) \subset \mathcal{P}_2(\mathbb{X})$ R-CDT is bounded with
  $$\mathrm{std}(\widehat{\mathcal{R}_{\phi,\theta}[\mu]}) \geq c > 0 \quad \text{f.a.} \quad \theta \in \Theta.$$
- **Normalize** per slice:
  $$\mathcal{N}_{\phi,\theta}[\mu](t) = \frac{\widehat{\mathcal{R}_{\phi,\theta}[\mu]}(t) - \mathrm{mean}(\widehat{\mathcal{R}_{\phi,\theta}[\mu]})}{\mathrm{std}(\widehat{\mathcal{R}_{\phi,\theta}[\mu]})}$$
- **Sorting** the directions $\theta \in \Theta$ via:
  $$h : L^{\infty}(\Theta) \to \mathbb{R}$$
- **Define** $_{\mathrm{h}}$NR-CDT: $h\big(\mathcal{N}_{\phi}[\mu](t,\cdot)\big)$

Radon on Measures & Core Idea

- Restricted Radon transfrom via measurable $\phi : \mathbb{X} \times \Theta \to \mathbb{R}$
$$\mathcal{R}_{\phi,\theta} : \mathcal{M}(\mathbb{X}) \to \mathcal{M}(\mathbb{R})$$
$$\mu \mapsto (S_{\phi,\theta})_{\#}\mu, \; S_{\phi,\theta} \coloneqq \phi(\cdot, \boldsymbol{\theta})$$
- Slice per direction
  $$\mu \in \mathcal{P}(\mathbb{X}) \quad \Rightarrow \quad \mathcal{R}_{\phi,\theta}[\mu] \in \mathcal{P}(\mathbb{R})$$
- 1D embedding via CDT, wrt. ref ρ
$$\widehat{\mu} = \operatorname*{arg\,min}_{T_{\#}\rho = \mu} \int_{\mathbb{R}} c(s - T(s)) \,\mathrm{d}\rho(s), \quad \text{atomless } \rho, \mu \in \mathcal{P}(\mathbb{R})$$
- R-CDT: collect sliced CDTs
  $$\widehat{\mathcal{R}_{\phi,\theta}[\mu]} \in \mathrm{L}^2_{\rho}(\mathbb{R}) \text{ for } \mu \in \mathcal{P}_2(\mathbb{X})$$

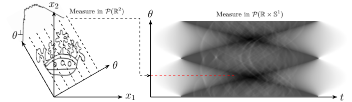
Theoretical Results

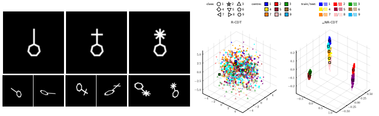
- **Translation:** for suitable $(\mathbb{X}, \Theta, \phi)$
  $$\widehat{\mathcal{R}_{\phi}[T_{\#}\mu]} = \hat{T} \circ \widehat{\mathcal{R}_{\phi}[\mu]}$$
  for specific T correspond to $\hat{T}$

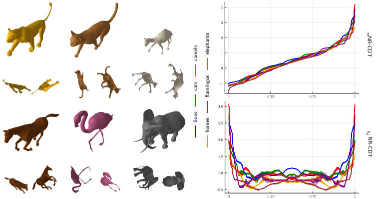
i) **Invariance:** $\mathcal{N}_{h,\phi}[T_{\#}\mu] = \mathcal{N}_{h,\phi}[\mu]$

ii) **Singleton sets** in $_{\mathrm{h}}$NR-CDT-space
$$\mathbb{F}_{\mu} = \{T_{\#}\mu\} \xrightarrow{{}_{\mathrm{h}}\text{NR-CDT}} \mathcal{N}_{h,\phi}[\mathbb{F}_{\mu}] = \mathcal{N}_{h,\phi}[\mu]$$

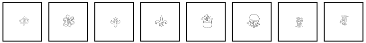
iii) **Robustness** of $_{\mathrm{m}}$NR-CDT and $_{\mathrm{a}}$NR-CDT in Wasserstein-spaces
$$h \in \left\{\sup\nolimits_{\boldsymbol{\theta}\in\Theta}, \int_{\Theta}\right\}.$$







# classes	method	NT # angles 8	16	32	64	128	1-NN # training samples 5	10
100	Euclidean	0.0096					0.0202 ± 0.0021	0.0242 ± 0.0022
	R-CDT	0.0192	0.0188	0.0190	0.0190	0.0192	0.0178 ± 0.0022	0.0210 ± 0.0024
	$_{\mathrm{m}}$NR-CDT	**0.4952**	**0.9032**	**0.9922**	**0.9992**	**1.0000**	$\mathbf{1.0000 \pm 0.0000}$	$\mathbf{1.0000 \pm 0.0000}$
1000	$_{\mathrm{m}}$NR-CDT	**0.3433**	**0.8256**	**0.9842**	**0.9977**	**1.0000**	$\mathbf{0.9987 \pm 0.0001}$	$\mathbf{0.9997 \pm 0.0001}$



# angles	8		16		32		64		128	
	$\lVert\cdot\rVert_\infty$	$\lVert\cdot\rVert_2$	$\lVert\cdot\rVert_\infty$	$\lVert\cdot\rVert_2$	$\lVert\cdot\rVert_\infty$	$\lVert\cdot\rVert_2$	$\lVert\cdot\rVert_\infty$	$\lVert\cdot\rVert_2$	$\lVert\cdot\rVert_\infty$	$\lVert\cdot\rVert_2$
R-CDT	0.4285	0.1785	0.3928	0.1785	0.4642	0.1785	0.4285	0.1785	0.4285	0.1785
$_{\mathrm{m}}$NR-CDT	0.8571	0.9642	0.9285	**1.0000**	0.9285	**1.0000**	0.9295	0.9642	0.9285	0.9642
$_{\mathrm{a}}$NR-CDT	0.9642	0.8571	**1.0000**	0.9642	**1.0000**	0.9642	**1.0000**	0.9642	**1.0000**	0.9642
Eucl.	0.2857	0.1785								