Denoising Techniques for Manifold Valued Data – Relaxations

Gabriele Steidl

Robert Beinert

Miniworkshop 2550b – MFO 2025, December 7th–12th — Oberwolfach, Germany

Jonas Bresch
TU Berlin, Institut für Mathematik

December 11, 2025

- B.Sc. and M.Sc. in pure math (group theory) at TU Braunschweig (2017—2022).
- Field of interests: p-groups and the classification of those, using invariant subgroups, e.g., the Schur multiplier and the Frattini subgroup.
- Ph.D. Position in applied mathematics at TU Berlin (since Dec. 16th, 2022).
- Phase II Member Berlin Mathematical School (since 2025).
- Field of interests (nowadays):
  - image analysis using different manifolds,
  - optimal transport—e.g., regularization techniques,
  - Radon transformation and different CDTs for classification on small data-regimes,
  - stochastic, randomized linear algebra—e.g., computation of the operator quantities.

- **Focus:** Important variational model with quadratic TIKHONOV and TOTAL VARIATION (TV) regularization.

[1] R. Beinert, J. Bresch, and G. Steidl. In: *Inverse Probl. Imaging* 19.1 (2025), pp. 87–108.
[2] R. Beinert and J. Bresch. In: *Proc. Appl. Math. Mech.* 24.1 (2024), e202400016.
[3] R. Beinert and J. Bresch. In: vol. 15667. Lect. Notes Comput. Sci. 2. Dartington, UK: Springer, 2025, pp. 16–29.
[4] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- **Focus:** Important variational model with quadratic TIKHONOV and TOTAL VARIATION (TV) regularization.
- **Property of such solutions:** Values are close to each other ((un-)smoothly).

[1] R. Beinert, J. Bresch, and G. Steidl. In: *Inverse Probl. Imaging* 19.1 (2025), pp. 87–108.
[2] R. Beinert and J. Bresch. In: *Proc. Appl. Math. Mech.* 24.1 (2024), e202400016.
[3] R. Beinert and J. Bresch. In: vol. 15667. Lect. Notes Comput. Sci. 2. Dartington, UK: Springer, 2025, pp. 16–29.
[4] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- **Focus:** Important variational model with quadratic TIKHONOV and TOTAL VARIATION (TV) regularization.
- **Property of such solutions:** Values are close to each other ((un-)smoothly).
- **Disadvantage:** Difficult nonconvex problem, due to the nonconvex manifold constraints.

[1] R. Beinert, J. Bresch, and G. Steidl. In: *Inverse Probl. Imaging* 19.1 (2025), pp. 87–108.

[2] R. Beinert and J. Bresch. In: *Proc. Appl. Math. Mech.* 24.1 (2024), e202400016.

[3] R. Beinert and J. Bresch. In: vol. 15667. Lect. Notes Comput. Sci. 2. Dartington, UK: Springer, 2025, pp. 16–29.

[4] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- **Focus:** Important variational model with quadratic Tikhonov and Total Variation (TV) regularization.
- **Property of such solutions:** Values are close to each other ((un-)smoothly).
- **Disadvantage:** Difficult nonconvex problem, due to the nonconvex manifold constraints.
- **Done 1):** Nicely simplified relaxed model without loosing any information for Tikhonov[1] on $\mathbb{S}_1$ with generalizations to $\mathbb{S}_{d-1}$ and SO(3).

[1] R. Beinert, J. Bresch, and G. Steidl. In: *Inverse Probl. Imaging* 19.1 (2025), pp. 87–108.
[2] R. Beinert and J. Bresch. In: *Proc. Appl. Math. Mech.* 24.1 (2024), e202400016.
[3] R. Beinert and J. Bresch. In: vol. 15667. Lect. Notes Comput. Sci. 2. Dartington, UK: Springer, 2025, pp. 16–29.
[4] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- **Focus:** Important variational model with quadratic Tikhonov and Total Variation (TV) regularization.
- **Property of such solutions:** Values are close to each other ((un-)smoothly).
- **Disadvantage:** Difficult nonconvex problem, due to the nonconvex manifold constraints.
- **Done 1):** Nicely simplified relaxed model without loosing any information for Tikhonov[1] on $\mathbb{S}_1$ with generalizations to $\mathbb{S}_{d-1}$ and $\mathrm{SO}(3)$.
- **Done 2):** Improvement for the TV[2], w.r.t. geodesic models, e.g. for the $\mathbb{S}_0$—with "tightness" and applications for any sphere, and generalization to the $\mathbb{S}_0^d$.

[1] R. Beinert, J. Bresch, and G. Steidl. In: *Inverse Probl. Imaging* 19.1 (2025), pp. 87–108.
[2] R. Beinert and J. Bresch. In: *Proc. Appl. Math. Mech.* 24.1 (2024), e202400016.
[3] R. Beinert and J. Bresch. In: vol. 15667. Lect. Notes Comput. Sci. 2. Dartington, UK: Springer, 2025, pp. 16–29.
[4] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- **Focus:** Important variational model with quadratic TIKHONOV and TOTAL VARIATION (TV) regularization.
- **Property of such solutions:** Values are close to each other ((un-)smoothly).
- **Disadvantage:** Difficult nonconvex problem, due to the nonconvex manifold constraints.
- **Done 1):** Nicely simplified relaxed model without loosing any information for TIKHONOV[1] on $\mathbb{S}_1$ with generalizations to $\mathbb{S}_{d-1}$ and SO(3).
- **Done 2):** Improvement for the TV[2], w.r.t. geodesic models, e.g. for the $\mathbb{S}_0$—with "tightness" and applications for any sphere, and generalization to the $\mathbb{S}_0^d$.
- **Done 3):** Application to more complicated problems, i.e., non-convex manifolds, for instance:

[1] R. Beinert, J. Bresch, and G. Steidl. In: *Inverse Probl. Imaging* 19.1 (2025), pp. 87–108.

[2] R. Beinert and J. Bresch. In: *Proc. Appl. Math. Mech.* 24.1 (2024), e202400016.

[3] R. Beinert and J. Bresch. In: vol. 15667. Lect. Notes Comput. Sci. 2. Dartington, UK: Springer, 2025, pp. 16–29.

[4] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- **Focus:** Important variational model with quadratic Tikhonov and Total Variation (TV) regularization.
- **Property of such solutions:** Values are close to each other ((un-)smoothly).
- **Disadvantage:** Difficult nonconvex problem, due to the nonconvex manifold constraints.
- **Done 1):** Nicely simplified relaxed model without loosing any information for Tikhonov[1] on $\mathbb{S}_1$ with generalizations to $\mathbb{S}_{d-1}$ and SO(3).
- **Done 2):** Improvement for the TV[2], w.r.t. geodesic models, e.g. for the $\mathbb{S}_0$—with "tightness" and applications for any sphere, and generalization to the $\mathbb{S}_0^d$.
- **Done 3):** Application to more complicated problems, i.e., non-convex manifolds, for instance:
  - the hyperboloid $\mathbb{H}_d := \{\mathbb{R}^{d+1} \mid \boldsymbol{x}_1^2 + \ldots + \boldsymbol{x}_d^2 = \boldsymbol{x}_{d+1}^2 - 1, \boldsymbol{x}_{d+1} > 0\}$ with application[3] for the Pioncare ball and space.

[1] R. Beinert, J. Bresch, and G. Steidl. In: *Inverse Probl. Imaging* 19.1 (2025), pp. 87–108.
[2] R. Beinert and J. Bresch. In: *Proc. Appl. Math. Mech.* 24.1 (2024), e202400016.
[3] R. Beinert and J. Bresch. In: vol. 15667. Lect. Notes Comput. Sci. 2. Dartington, UK: Springer, 2025, pp. 16–29.
[4] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- **Focus:** Important variational model with quadratic TIKHONOV and TOTAL VARIATION (TV) regularization.
- **Property of such solutions:** Values are close to each other ((un-)smoothly).
- **Disadvantage:** Difficult nonconvex problem, due to the nonconvex manifold constraints.
- **Done 1):** Nicely simplified relaxed model without loosing any information for TIKHONOV[1] on $\mathbb{S}_1$ with generalizations to $\mathbb{S}_{d-1}$ and SO(3).
- **Done 2):** Improvement for the TV[2], w.r.t. geodesic models, e.g. for the $\mathbb{S}_0$—with "tightness" and applications for any sphere, and generalization to the $\mathbb{S}_0^d$.
- **Done 3):** Application to more complicated problems, i.e., non-convex manifolds, for instance:
  - the hyperboloid $\mathbb{H}_d := \{\mathbb{R}^{d+1} \mid \boldsymbol{x}_1^2 + \ldots + \boldsymbol{x}_d^2 = \boldsymbol{x}_{d+1}^2 - 1, \boldsymbol{x}_{d+1} > 0\}$ with application[3] for the PIONCARE ball and space.
  - the STIEFEL manifold[4] $\mathbb{V}_d(\ell) := \{\boldsymbol{V} \in \mathbb{R}^{d \times \ell} \mid \boldsymbol{V}^* \boldsymbol{V} = I_\ell\}$.

[1] R. Beinert, J. Bresch, and G. Steidl. In: *Inverse Probl. Imaging* 19.1 (2025), pp. 87–108.
[2] R. Beinert and J. Bresch. In: *Proc. Appl. Math. Mech.* 24.1 (2024), e202400016.
[3] R. Beinert and J. Bresch. In: vol. 15667. Lect. Notes Comput. Sci. 2. Dartington, UK: Springer, 2025, pp. 16–29.
[4] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- **Focus:** Important variational model with quadratic TIKHONOV and TOTAL VARIATION (TV) regularization.
- **Property of such solutions:** Values are close to each other ((un-)smoothly).
- **Disadvantage:** Difficult nonconvex problem, due to the nonconvex manifold constraints.
- **Done 1):** Nicely simplified relaxed model without loosing any information for TIKHONOV[1] on $\mathbb{S}_1$ with generalizations to $\mathbb{S}_{d-1}$ and SO(3).
- **Done 2):** Improvement for the TV[2], w.r.t. geodesic models, e.g. for the $\mathbb{S}_0$—with "tightness" and applications for any sphere, and generalization to the $\mathbb{S}_0^d$.
- **Done 3):** Application to more complicated problems, i.e., non-convex manifolds, for instance:
  - the hyperboloid $\mathbb{H}_d := \{\mathbb{R}^{d+1} \mid \boldsymbol{x}_1^2 + \ldots + \boldsymbol{x}_d^2 = \boldsymbol{x}_{d+1}^2 - 1, \boldsymbol{x}_{d+1} > 0\}$ with application[3] for the PIONCARE ball and space.
  - the STIEFEL manifold[4] $\mathbb{V}_d(\ell) := \{\boldsymbol{V} \in \mathbb{R}^{d \times \ell} \mid \boldsymbol{V}^* \boldsymbol{V} = I_\ell\}$.
- **Aim:** Overview and motivation for a bunch of other manifold.

[1] R. Beinert, J. Bresch, and G. Steidl. In: *Inverse Probl. Imaging* 19.1 (2025), pp. 87–108.
[2] R. Beinert and J. Bresch. In: *Proc. Appl. Math. Mech.* 24.1 (2024), e202400016.
[3] R. Beinert and J. Bresch. In: vol. 15667. Lect. Notes Comput. Sci. 2. Dartington, UK: Springer, 2025, pp. 16–29.
[4] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

– Let $G = (V, E)$ be a connected, undirected graph,

[5] Laurent Condat. In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782.

Definition

- Let $G = (V, E)$ be a connected, undirected graph,
- with vertices $V := \{1, \dots, N\}$, i.e. $N := |V|$,

[5] Laurent Condat. In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782.

- Let $G = (V, E)$ be a connected, undirected graph,
- with vertices $V \coloneqq \{1, \dots, N\}$, i.e. $N \coloneqq |V|$,
- and edges $E \subset \{(n, m) \in V \times V : n < m\}$,
  with $M \coloneqq |E|$.

[5] Laurent Condat. In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782.

- Let $G = (V, E)$ be a connected, undirected graph,
- with vertices $V \coloneqq \{1, \dots, N\}$, i.e. $N \coloneqq |V|$,
- and edges $E \subset \{(n, m) \in V \times V : n < m\}$,
  with $M \coloneqq |E|$.
- general denoising model, for instance G is the line- or grid-graph.

[5] Laurent Condat. In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782.

- Let $G = (V, E)$ be a connected, undirected graph,
- with vertices $V := \{1, \ldots, N\}$, i.e. $N := |V|$,
- and edges $E \subset \{(n, m) \in V \times V : n < m\}$, with $M := |E|$.
- general denoising model, for instance G is the line- or grid-graph.

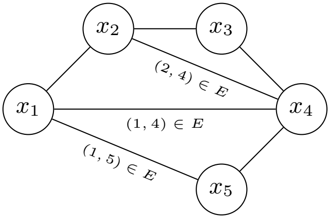


Figure 1: x_n manifold-valued, identified by $n \in V$ with adjacentcies given by $(n, m) \in E$.

[5] Laurent Condat. In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782.

- Let $G = (V, E)$ be a connected, undirected graph,
- with vertices $V \coloneqq \{1, \ldots, N\}$, i.e. $N \coloneqq |V|$,
- and edges $E \subset \{(n, m) \in V \times V : n < m\}$, with $M \coloneqq |E|$.
- general denoising model, for instance G is the line- or grid-graph.

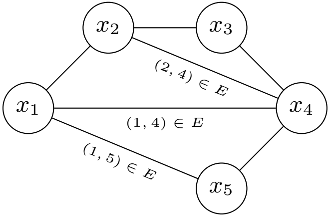


Figure 1: x_n manifold-valued, identified by $n \in V$ with adjacentcies given by $(n, m) \in E$.

- **Former approach by Condat (2022)**[5]**:** Denoise a disturbed $\mathbb{S}_1$-valued signal on G, i.e. $y = (y_n)_{n \in V} \in \mathbb{C}^N$ using Tikhonov and $\mathbb{S}_1 \hookrightarrow \mathbb{C}$.

[5] Laurent Condat. In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782.

- Let $G = (V, E)$ be a connected, undirected graph,
- with vertices $V := \{1, \ldots, N\}$, i.e. $N := |V|$,
- and edges $E \subset \{(n, m) \in V \times V : n < m\}$, with $M := |E|$.
- general denoising model, for instance G is the line- or grid-graph.

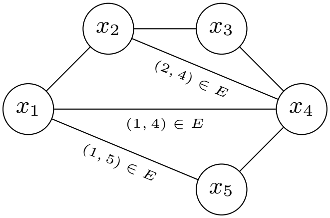


FIGURE 1: x_n manifold-valued, identified by $n \in V$ with adjacentcies given by $(n, m) \in E$.

- **Former approach by CONDAT (2022)[5]:** Denoise a disturbed $\mathbb{S}_1$-valued signal on G, i.e. $y = (y_n)_{n \in V} \in \mathbb{C}^N$ using TIKHONOV and $\mathbb{S}_1 \hookrightarrow \mathbb{C}$.
- **First approach:** $\mathbb{S}_1$ can be embedded into $\mathbb{R}^2$, denote $\boldsymbol{y} = (\boldsymbol{y}_n)_{n \in V} \in \mathbb{R}^{2N}$ and model can be simplified [1]. w.r.t. the matrix representation.

[5] Laurent Condat. In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782.

- Let $G = (V, E)$ be a connected, undirected graph,
- with vertices $V := \{1, \dots, N\}$, i.e. $N := |V|$,
- and edges $E \subset \{(n, m) \in V \times V : n < m\}$, with $M := |E|$.
- general denoising model, for instance G is the line- or grid-graph.

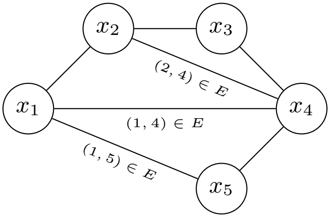


FIGURE 1: x_n manifold-valued, identified by $n \in V$ with adjacentcies given by $(n, m) \in E$.

- **Former approach by CONDAT (2022)[5]:** Denoise a disturbed $\mathbb{S}_1$-valued signal on G, i.e. $y = (y_n)_{n \in V} \in \mathbb{C}^N$ using TIKHONOV and $\mathbb{S}_1 \hookrightarrow \mathbb{C}$.
- **First approach:** $\mathbb{S}_1$ can be embedded into $\mathbb{R}^2$, denote $\boldsymbol{y} = (\boldsymbol{y}_n)_{n \in V} \in \mathbb{R}^{2N}$ and model can be simplified [1]. w.r.t. the matrix representation.
- **Second approach:** Theory can be generalized for spheres and SO(3)-valued data[1].

[5] Laurent Condat. In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782.

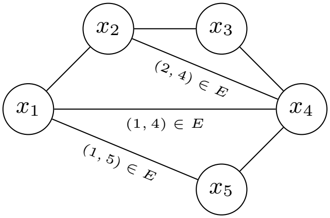


FIGURE 1: x_n manifold-valued, identified by $n \in V$ with adjacentcies given by $(n, m) \in E$.

- Let $G = (V, E)$ be a connected, undirected graph,
- with vertices $V := \{1, \dots, N\}$, i.e. $N := |V|$,
- and edges $E \subset \{(n, m) \in V \times V : n < m\}$, with $M := |E|$.
- general denoising model, for instance G is the line- or grid-graph.
- **Former approach by CONDAT (2022)[5]:** Denoise a disturbed $\mathbb{S}_1$-valued signal on G, i.e. $y = (y_n)_{n \in V} \in \mathbb{C}^N$ using TIKHONOV and $\mathbb{S}_1 \hookrightarrow \mathbb{C}$.
- **First approach:** $\mathbb{S}_1$ can be embedded into $\mathbb{R}^2$, denote $\boldsymbol{y} = (\boldsymbol{y}_n)_{n \in V} \in \mathbb{R}^{2N}$ and model can be simplified [1]. w.r.t. the matrix representation.
- **Second approach:** Theory can be generalized for spheres and SO(3)-valued data[1].
- **Third approach:** Application for TV regularization[2][3].

[5] Laurent Condat. In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782.

- Let $G = (V, E)$ be a connected, undirected graph,
- with vertices $V := \{1, \dots, N\}$, i.e. $N := |V|$,
- and edges $E \subset \{(n,m) \in V \times V : n < m\}$, with $M := |E|$.
- general denoising model, for instance G is the line- or grid-graph.

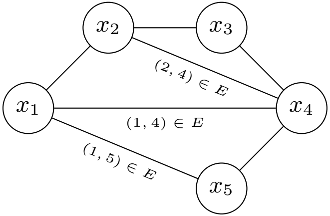


Figure 1: x_n manifold-valued, identified by $n \in V$ with adjacentcies given by $(n,m) \in E$.

- **Former approach by Condat (2022)[5]:** Denoise a disturbed $\mathbb{S}_1$-valued signal on G, i.e. $y = (y_n)_{n \in V} \in \mathbb{C}^N$ using Tikhonov and $\mathbb{S}_1 \hookrightarrow \mathbb{C}$.
- **First approach:** $\mathbb{S}_1$ can be embedded into $\mathbb{R}^2$, denote $\boldsymbol{y} = (\boldsymbol{y}_n)_{n \in V} \in \mathbb{R}^{2N}$ and model can be simplified [1]. w.r.t. the matrix representation.
- **Second approach:** Theory can be generalized for spheres and SO(3)-valued data[1].
- **Third approach:** Application for TV regularization[2][3].
- **Forth approach:** Generalization for TV and Tikhonov regularization on other manifolds[3][4].

[5] Laurent Condat. In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782.

– Let $\mathbb{S}_{\mathbb{C}} := \{z \in \mathbb{C} : |z| = 1\}$ denote the (complex-valued) unit circle.

- Let $\mathbb{S}_\mathbb{C} \coloneqq \{z \in \mathbb{C} : |z| = 1\}$ denote the (complex-valued) unit circle.
- **Aim:** recover a signal $x \coloneqq (x_n)_{n \in V} \in \mathbb{S}_\mathbb{C}^N$ on G from noisy measurements $y \coloneqq (y_n)_{n \in V} \in {\mathbb{S}_\mathbb{C}}^N$.

- Let $\mathbb{S}_{\mathbb{C}} := \{z \in \mathbb{C} : |z| = 1\}$ denote the (complex-valued) unit circle.
- **Aim:** recover a signal $x := (x_n)_{n \in V} \in \mathbb{S}_{\mathbb{C}}^N$ on G from noisy measurements $y := (y_n)_{n \in V} \in {\mathbb{S}_{\mathbb{C}}}^N$.
- **Searching:** minimizer of the Tikhonov-like regularized functional (**nonconvex**)

$$\operatorname*{arg\,min}_{x \in \mathbb{S}_{\mathbb{C}}^N} \sum_{n \in V} \frac{w_n}{2} \left|x_n - y_n\right|^2 + \sum_{(n,m) \in E} \frac{\lambda_{(n,m)}}{2} \left|x_n - x_m\right|^2, \qquad (\star)$$

where $w := (w_n)_{n \in V} \in \mathbb{R}_+^N$ and $\lambda := (\lambda_{(n,m)})_{(n,m) \in E} \in \mathbb{R}_+^M$ are positive weights.

- Let $\mathbb{S}_{\mathbb{C}} := \{z \in \mathbb{C} : |z| = 1\}$ denote the (complex-valued) unit circle.
- **Aim:** recover a signal $x := (x_n)_{n \in V} \in \mathbb{S}_{\mathbb{C}}^N$ on G from noisy measurements $y := (y_n)_{n \in V} \in {\mathbb{S}_{\mathbb{C}}}^N$.
- **Searching:** minimizer of the Tikhonov-like regularized functional (**nonconvex**)

$$\operatorname*{arg\,min}_{x \in \mathbb{S}_{\mathbb{C}}^N} \sum_{n \in V} \frac{w_n}{2} |x_n - y_n|^2 + \sum_{(n,m) \in E} \frac{\lambda_{(n,m)}}{2} |x_n - x_m|^2, \qquad (\star)$$

  where $w := (w_n)_{n \in V} \in \mathbb{R}_+^N$ and $\lambda := (\lambda_{(n,m)})_{(n,m) \in E} \in \mathbb{R}_+^M$ are positive weights.
- simple equation: $|x_n - y_n|^2 = |x_n|^2 - 2\Re[x_n \bar{y}_n] + |y_n|^2$ for $x_n, y_n \in \mathbb{C}$.

- Let $\mathbb{S}_{\mathbb{C}} \coloneqq \{z \in \mathbb{C} : |z| = 1\}$ denote the (complex-valued) unit circle.
- **Aim:** recover a signal $x \coloneqq (x_n)_{n \in V} \in \mathbb{S}_{\mathbb{C}}^N$ on G from noisy measurements $y \coloneqq (y_n)_{n \in V} \in {\mathbb{S}_{\mathbb{C}}}^N$.
- **Searching:** minimizer of the Tikhonov-like regularized functional (**nonconvex**)

$$\operatorname*{arg\,min}_{x \in \mathbb{S}_{\mathbb{C}}^N} \sum_{n \in V} \frac{w_n}{2} |x_n - y_n|^2 + \sum_{(n,m) \in E} \frac{\lambda_{(n,m)}}{2} |x_n - x_m|^2, \tag{$\star$}$$

  where $w \coloneqq (w_n)_{n \in V} \in \mathbb{R}_+^N$ and $\lambda \coloneqq (\lambda_{(n,m)})_{(n,m) \in E} \in \mathbb{R}_+^M$ are positive weights.
- simple equation: $|x_n - y_n|^2 = |x_n|^2 - 2\Re[x_n \bar{y}_n] + |y_n|^2$ for $x_n, y_n \in \mathbb{C}$.
- **Exploiting:** $|x_n| = 1$, and rewriting the **nonconvex problem** as

$$(\star) = \operatorname*{arg\,max}_{x \in \mathbb{S}_{\mathbb{C}}^N} \sum_{n \in V} w_n \Re[x_n \bar{y}_n] + \sum_{(n,m) \in E} \lambda_{(n,m)} \Re[\bar{x}_m x_n].$$

- **Introducing:** $r := (r_{(n,m)})_{(n,m)\in E} \in \mathbb{C}^M$, and rearranging the latter (**nonconvex**) problem into

$$(\star) = \operatorname*{arg\,min}_{x\in\mathbb{S}_{\mathbb{C}}^N, r\in\mathbb{C}^M} \mathcal{J}(x,r) \;\text{ s.t. }\; r_{(n,m)} = \bar{x}_m x_n \;\forall\, (n,m)\in E$$

with

$$\mathcal{J}(x,r) := -\sum_{n\in V} w_n\, \Re[x_n \bar{y}_n] - \sum_{(n,m)\in E} \lambda_{(n,m)}\, \Re[r_{(n,m)}].$$

- **Introducing:** $r := (r_{(n,m)})_{(n,m)\in E} \in \mathbb{C}^M$, and rearranging the latter (**nonconvex**) problem into

$$(\star) = \operatorname*{arg\,min}_{x\in\mathbb{S}_{\mathbb{C}}^N, r\in\mathbb{C}^M} \mathcal{J}(x,r) \;\text{ s.t. }\; r_{(n,m)} = \bar{x}_m x_n \;\forall\, (n,m)\in E$$

  with

$$\mathcal{J}(x,r) := -\sum_{n\in V} w_n\, \Re[x_n \bar{y}_n] - \sum_{(n,m)\in E} \lambda_{(n,m)}\, \Re[r_{(n,m)}].$$

- **Core idea:** Condat's convex relaxation[5] encodes (**nonconvex**) $x \in \mathbb{S}_{\mathbb{C}}^N$ as follows:

- **Introducing:** $r := (r_{(n,m)})_{(n,m)\in E} \in \mathbb{C}^M$, and rearranging the latter (**nonconvex**) problem into

$$(\star) = \operatorname*{arg\,min}_{x\in\mathbb{S}_{\mathbb{C}}^N, r\in\mathbb{C}^M} \mathcal{J}(x,r) \;\text{ s.t. }\; r_{(n,m)} = \bar{x}_m x_n \;\forall\, (n,m) \in E$$

  with

$$\mathcal{J}(x,r) := -\sum_{n\in V} w_n \,\Re[x_n \bar{y}_n] - \sum_{(n,m)\in E} \lambda_{(n,m)} \,\Re[r_{(n,m)}].$$

- **Core idea:** Condat's convex relaxation[5] encodes (**nonconvex**) $x \in \mathbb{S}_{\mathbb{C}}^N$ as follows:

Lemma 1 (Beinert, B., Steidl[1])

Let $n, m \in V$ and $(n,m) \in E$. Then $x_n, x_m \in \mathbb{S}_{\mathbb{C}}$ and $r_{(n,m)} = \bar{x}_m x_n$ if and only if

$$P_{(n,m)} := \begin{bmatrix} 1 & x_n & x_m \\ \bar{x}_n & 1 & \bar{r}_{(n,m)} \\ \bar{x}_m & r_{(n,m)} & 1 \end{bmatrix} \in \mathbb{C}^{3\times 3}$$

is positive semi-definite and has rank one.

– Motivated by parameterization of the circle valued signal by measures on $\mathbb{S}^2_{\mathbb{C}}$, using 2d Diracs on $\mathbb{S}_{\mathbb{C}}$,

[6] R. Meziat. In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324.

[7] C. Carathéodory. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217.

- Motivated by parameterization of the circle valued signal by measures on $\mathbb{S}^2_{\mathbb{C}}$, using 2d Diracs on $\mathbb{S}_{\mathbb{C}}$,
- Like in the *methods of moments*[6] going back to[7],

[6] R. Meziat. In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324.
[7] C. Carathéodory. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217.

- Motivated by parameterization of the circle valued signal by measures on $\mathbb{S}^2_{\mathbb{C}}$, using 2d Diracs on $\mathbb{S}_{\mathbb{C}}$,
- Like in the *methods of moments*[6] going back to[7],
- Selection of finite number of their Fourier coefficients for parametrization for definition of the *matrix of moments* $\rightarrow$ method is called *Fourier lifting*.

[6] R. Meziat. In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324.

[7] C. Carathéodory. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217.

- Motivated by parameterization of the circle valued signal by measures on $\mathbb{S}^2_{\mathbb{C}}$, using 2d Diracs on $\mathbb{S}_{\mathbb{C}}$,
- Like in the *methods of moments*[6] going back to[7],
- Selection of finite number of their Fourier coefficients for parametrization for definition of the *matrix of moments* $\to$ method is called *Fourier lifting*.

Proof of Lemma 1 (Beinert, B., Steidl[1]).

$\square$

---

[6] R. Meziat. In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324.

[7] C. Carathéodory. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217.

– Motivated by parameterization of the circle valued signal by measures on $\mathbb{S}^2_{\mathbb{C}}$, using 2d Diracs on $\mathbb{S}_{\mathbb{C}}$,

– Like in the *methods of moments*[6] going back to[7],

– Selection of finite number of their Fourier coefficients for parametrization for definition of the *matrix of moments* $\to$ method is called *Fourier lifting*.

Proof of Lemma 1 (Beinert, B., Steidl[1]).

- Let $x_n, x_m \in \mathbb{S}_{\mathbb{C}}$ and $r_{(n,m)} = \bar{x}_m x_n$,

$\square$

[6] R. Meziat. In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324.

[7] C. Carathéodory. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217.

– Motivated by parameterization of the circle valued signal by measures on $\mathbb{S}_{\mathbb{C}}^2$, using 2d Diracs on $\mathbb{S}_{\mathbb{C}}$,

– Like in the *methods of moments*[6] going back to[7],

– Selection of finite number of their Fourier coefficients for parametrization for definition of the *matrix of moments* $\to$ method is called *Fourier lifting*.

Proof of Lemma 1 (Beinert, B., Steidl[1]).

- Let $x_n, x_m \in \mathbb{S}_{\mathbb{C}}$ and $r_{(n,m)} = \bar{x}_m x_n$,
- Then $P_{(n,m)} = (1, x_n, x_m)^{\mathrm{H}}(1, x_n, x_m)$ has rank one and is positive semi-definite.

$\square$

[6] R. Meziat. In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324.

[7] C. Carathéodory. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217.

– Motivated by parameterization of the circle valued signal by measures on $\mathbb{S}_{\mathbb{C}}^2$, using 2d Diracs on $\mathbb{S}_{\mathbb{C}}$,
– Like in the *methods of moments*[6] going back to[7],
– Selection of finite number of their Fourier coefficients for parametrization for definition of the *matrix of moments* → method is called *Fourier lifting*.

Proof of Lemma 1 (Beinert, B., Steidl[1]).

- Let $x_n, x_m \in \mathbb{S}_{\mathbb{C}}$ and $r_{(n,m)} = \bar{x}_m x_n$,
- Then $P_{(n,m)} = (1, x_n, x_m)^{\mathrm{H}}(1, x_n, x_m)$ has rank one and is positive semi-definite.
- Conversely, positive semi-definitness and the rank-one condition imply that $P_{(n,m)}$ has the form

□

[6] R. Meziat. In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324.
[7] C. Carathéodory. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217.

- Motivated by parameterization of the circle valued signal by measures on $\mathbb{S}^2_{\mathbb{C}}$, using 2d Diracs on $\mathbb{S}_{\mathbb{C}}$,
- Like in the *methods of moments*[6] going back to[7],
- Selection of finite number of their Fourier coefficients for parametrization for definition of the *matrix of moments* $\rightarrow$ method is called *Fourier lifting*.

Proof of Lemma 1 (Beinert, B., Steidl[1]).

- Let $x_n, x_m \in \mathbb{S}_{\mathbb{C}}$ and $r_{(n,m)} = \bar{x}_m x_n$,
- Then $P_{(n,m)} = (1, x_n, x_m)^{\mathrm{H}}(1, x_n, x_m)$ has rank one and is positive semi-definite.
- Conversely, positive semi-definitness and the rank-one condition imply that $P_{(n,m)}$ has the form

$$P_{(n,m)} = (a,b,c)^{\mathrm{H}}(a,b,c) = \begin{bmatrix} \bar{a}a & \bar{a}b & \bar{a}c \\ \bar{b}a & \bar{b}b & \bar{b}a \\ \bar{c}a & \bar{c}b & \bar{c}c \end{bmatrix} \qquad \text{for some} \qquad a,b,c \in \mathbb{C}.$$

□

[6] R. Meziat. In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324.

[7] C. Carathéodory. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217.

- Motivated by parameterization of the circle valued signal by measures on $\mathbb{S}^2_{\mathbb{C}}$, using 2d Diracs on $\mathbb{S}_{\mathbb{C}}$,
- Like in the *methods of moments*[6] going back to[7],
- Selection of finite number of their Fourier coefficients for parametrization for definition of the *matrix of moments* $\rightarrow$ method is called *Fourier lifting*.

Proof of Lemma 1 (Beinert, B., Steidl[1]).

- Let $x_n, x_m \in \mathbb{S}_{\mathbb{C}}$ and $r_{(n,m)} = \bar{x}_m x_n$,
- Then $P_{(n,m)} = (1, x_n, x_m)^{\mathrm{H}}(1, x_n, x_m)$ has rank one and is positive semi-definite.
- Conversely, positive semi-definitness and the rank-one condition imply that $P_{(n,m)}$ has the form

$$P_{(n,m)} = (a,b,c)^{\mathrm{H}}(a,b,c) = \begin{bmatrix} \bar{a}a & \bar{a}b & \bar{a}c \\ \bar{b}a & \bar{b}b & \bar{b}a \\ \bar{c}a & \bar{c}b & \bar{c}c \end{bmatrix} \qquad \text{for some} \qquad a,b,c \in \mathbb{C}.$$

- Comparison with the assumption yields $|a| = |b| = |c| = 1$

$\square$

[6] R. Meziat. In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324.
[7] C. Carathéodory. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217.

- Motivated by parameterization of the circle valued signal by measures on $\mathbb{S}_{\mathbb{C}}^2$, using 2d Diracs on $\mathbb{S}_{\mathbb{C}}$,
- Like in the *methods of moments*[6] going back to[7],
- Selection of finite number of their Fourier coefficients for parametrization for definition of the *matrix of moments* $\rightarrow$ method is called *Fourier lifting*.

Proof of Lemma 1 (Beinert, B., Steidl[1]).

- Let $x_n, x_m \in \mathbb{S}_{\mathbb{C}}$ and $r_{(n,m)} = \bar{x}_m x_n$,
- Then $P_{(n,m)} = (1, x_n, x_m)^{\mathrm{H}}(1, x_n, x_m)$ has rank one and is positive semi-definite.
- Conversely, positive semi-definitness and the rank-one condition imply that $P_{(n,m)}$ has the form

$$P_{(n,m)} = (a,b,c)^{\mathrm{H}}(a,b,c) = \begin{bmatrix} \bar{a}a & \bar{a}b & \bar{a}c \\ \bar{b}a & \bar{b}b & \bar{b}a \\ \bar{c}a & \bar{c}b & \bar{c}c \end{bmatrix} \qquad \text{for some} \qquad a,b,c \in \mathbb{C}.$$

- Comparison with the assumption yields $|a| = |b| = |c| = 1$
- Thus $|x_n| = |\bar{a}||b| = 1$, $|x_m| = |\bar{a}||c| = 1$ as well as $r_{(n,m)} = \bar{c}b = \bar{c}a\bar{a}b = \bar{x}_m x_n$. $\square$

[6] R. Meziat. In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324.
[7] C. Carathéodory. In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217.

– Based on Lemma 1, the **nonconvex problem** becomes

$$(\star) = \operatorname*{arg\,min}_{x\in\mathbb{C}^N, r\in\mathbb{C}^M} \mathcal{J}(x,r) \text{ s.t. } P_{(n,m)} \succeq 0 \text{ and } \operatorname{rk}(P_{(n,m)}) = 1 \quad \forall (n,m) \in E$$

– Based on Lemma 1, the **nonconvex problem** becomes

$$(\star) = \operatorname*{arg\,min}_{x \in \mathbb{C}^N, r \in \mathbb{C}^M} \mathcal{J}(x,r) \text{ s.t. } P_{(n,m)} \succeq 0 \text{ and } \operatorname{rk}(P_{(n,m)}) = 1 \quad \forall (n,m) \in E$$

– **Neglecting:** The rank-one constraint, Condat (2022)[5] proposes to solve the relaxation:

relaxed complex model

$$\operatorname*{arg\,min}_{x \in \mathbb{C}^N, r \in \mathbb{C}^M} \mathcal{J}(x,r) \quad \text{s.t.} \quad P_{(n,m)} \succeq 0 \quad \forall (n,m) \in E.$$

- Based on Lemma 1, the **nonconvex problem** becomes

$$(\star) = \operatorname*{arg\,min}_{x\in\mathbb{C}^N, r\in\mathbb{C}^M} \mathcal{J}(x,r) \text{ s.t. } P_{(n,m)} \succeq 0 \text{ and } \operatorname{rk}(P_{(n,m)}) = 1 \quad \forall (n,m) \in E$$

- **Neglecting:** The rank-one constraint, Condat (2022)[5] proposes to solve the relaxation:

relaxed complex model

$$\operatorname*{arg\,min}_{x\in\mathbb{C}^N, r\in\mathbb{C}^M} \mathcal{J}(x,r) \quad \text{s.t.} \quad P_{(n,m)} \succeq 0 \quad \forall (n,m) \in E.$$

- **Now:** The **relaxed complex model** is a convex optimization problem, we can apply standard methods from convex analysis.

- **Aim:** Generalization via $\mathbb{R}^2 \hookleftarrow \mathbb{S}_1 := \{\boldsymbol{z} \in \mathbb{R}^2 : \|\boldsymbol{z}\| = 1\} \simeq \mathbb{S}_{\mathbb{C}}$.

– **Aim:** Generalization via $\mathbb{R}^2 \hookleftarrow \mathbb{S}_1 := \{\boldsymbol{z} \in \mathbb{R}^2 : \|\boldsymbol{z}\| = 1\} \simeq \mathbb{S}_{\mathbb{C}}$.

– We aim to solve the following **nonconvex** denoising problem

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{S}_1^N} \sum_{n \in V} \frac{w_n}{2} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \sum_{(n,m) \in E} \frac{\lambda_{(n,m)}}{2} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2. \tag{$\dagger$}$$

- **Aim:** Generalization via $\mathbb{R}^2 \hookleftarrow \mathbb{S}_1 := \{\boldsymbol{z} \in \mathbb{R}^2 : \|\boldsymbol{z}\| = 1\} \simeq \mathbb{S}_{\mathbb{C}}$.
- We aim to solve the following **nonconvex** denoising problem

$$\underset{\boldsymbol{x} \in \mathbb{S}_1^N}{\arg\min} \sum_{n \in V} \frac{w_n}{2} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \sum_{(n,m) \in E} \frac{\lambda_{(n,m)}}{2} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2. \tag{$\dagger$}$$

- Realizing the complex multiplication using the matrix representation of $z \in \mathbb{C}$ [3]

$$\boldsymbol{M}(z) := \boldsymbol{M}(\boldsymbol{z}) := \begin{bmatrix} \Re[\boldsymbol{z}] & -\Im[\boldsymbol{z}] \\ \Im[\boldsymbol{z}] & \Re[\boldsymbol{z}] \end{bmatrix} \qquad \Rightarrow \qquad \boldsymbol{M}(\boldsymbol{z})\boldsymbol{x} = \overrightarrow{zx} = \begin{bmatrix} \Re[zx] \\ \Im[zx] \end{bmatrix}.$$

- **Aim:** Generalization via $\mathbb{R}^2 \hookleftarrow \mathbb{S}_1 := \{\boldsymbol{z} \in \mathbb{R}^2 : \|\boldsymbol{z}\| = 1\} \simeq \mathbb{S}_{\mathbb{C}}$.
- We aim to solve the following **nonconvex** denoising problem

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_1^N} \sum_{n\in V} \frac{w_n}{2} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \sum_{(n,m)\in E} \frac{\lambda_{(n,m)}}{2} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2. \tag{†}$$

- Realizing the complex multiplication using the matrix representation of $z \in \mathbb{C}$ [3]

$$\boldsymbol{M}(z) := \boldsymbol{M}(\boldsymbol{z}) := \begin{bmatrix} \Re[\boldsymbol{z}] & -\Im[\boldsymbol{z}] \\ \Im[\boldsymbol{z}] & \Re[\boldsymbol{z}] \end{bmatrix} \qquad \Rightarrow \qquad \boldsymbol{M}(\boldsymbol{z})\boldsymbol{x} = \overrightarrow{zx} = \begin{bmatrix} \Re[zx] \\ \Im[zx] \end{bmatrix}.$$

- **Introducing:** $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n \in \mathbb{R}^2$, yields the real of the complex-valued version:

$$(\dagger) = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_1^N, \boldsymbol{r}\in(\mathbb{R}^2)^M} \mathcal{J}(\boldsymbol{x}, \boldsymbol{r}) \ \text{ s.t. } \ \boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n \quad \forall (n,m) \in E$$

with $\mathcal{J}(\boldsymbol{x}, \boldsymbol{r}) := -\sum_{n\in V} w_n \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle - \sum_{(n,m)\in E} \lambda_{(n,m)} \Re[\boldsymbol{r}_{(n,m)}]$.

Lemma 2 (Beinert, B., Steidl[1])

Let $n, m \in V$ and $(n,m) \in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_1$ and $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n$ if and only if the block matrix

$$\boldsymbol{P}_{(n,m)} := \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{x}_n) & \boldsymbol{M}(\boldsymbol{x}_m) \\ \boldsymbol{M}(\boldsymbol{x}_n)^{\mathrm{T}} & \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{r}_{(n,m)})^{\mathrm{T}} \\ \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} & \boldsymbol{M}(\boldsymbol{r}_{(n,m)}) & \boldsymbol{I}_2 \end{bmatrix} \in \mathbb{R}^{6 \times 6}$$

is positive semi-definite and has rank two.

LEMMA 2 (BEINERT, B., STEIDL[1])

Let $n, m \in V$ *and* $(n,m) \in E$. *Then* $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_1$ *and* $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n$ *if and only if the block matrix*

$$\boldsymbol{P}_{(n,m)} := \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{x}_n) & \boldsymbol{M}(\boldsymbol{x}_m) \\ \boldsymbol{M}(\boldsymbol{x}_n)^{\mathrm{T}} & \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{r}_{(n,m)})^{\mathrm{T}} \\ \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} & \boldsymbol{M}(\boldsymbol{r}_{(n,m)}) & \boldsymbol{I}_2 \end{bmatrix} \in \mathbb{R}^{6 \times 6}$$

is positive semi-definite and has rank two.

PROOF OF LEMMA 2 (BEINERT, B., STEIDL[1]).

□

Lemma 2 (Beinert, B., Steidl[1])

Let $n, m \in V$ and $(n,m) \in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_1$ and $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n$ if and only if the block matrix

$$\boldsymbol{P}_{(n,m)} := \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{x}_n) & \boldsymbol{M}(\boldsymbol{x}_m) \\ \boldsymbol{M}(\boldsymbol{x}_n)^{\mathrm{T}} & \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{r}_{(n,m)})^{\mathrm{T}} \\ \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} & \boldsymbol{M}(\boldsymbol{r}_{(n,m)}) & \boldsymbol{I}_2 \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

is positive semi-definite and has rank two.

Proof of Lemma 2 (Beinert, B., Steidl[1]).

- Using Schur's theorem to neglect the $\boldsymbol{M}(\cdot)$ and translate the positive semidefiniteness.

$\square$

Lemma 2 (Beinert, B., Steidl[1])

Let $n, m \in V$ and $(n,m) \in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_1$ and $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n$ if and only if the block matrix

$$\boldsymbol{P}_{(n,m)} := \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{x}_n) & \boldsymbol{M}(\boldsymbol{x}_m) \\ \boldsymbol{M}(\boldsymbol{x}_n)^{\mathrm{T}} & \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{r}_{(n,m)})^{\mathrm{T}} \\ \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} & \boldsymbol{M}(\boldsymbol{r}_{(n,m)}) & \boldsymbol{I}_2 \end{bmatrix} \in \mathbb{R}^{6 \times 6}$$

is positive semi-definite and has rank two.

Proof of Lemma 2 (Beinert, B., Steidl[1]).

- Using Schur's theorem to neglect the $\boldsymbol{M}(\cdot)$ and translate the positive semidefiniteness.
- The rank property, is, due to the $\boldsymbol{I}_2$, a "lowest"-rank property, such that the Schur complement is a full-zero block. □

LEMMA 2 (BEINERT, B., STEIDL[1])

Let $n, m \in V$ and $(n,m) \in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_1$ and $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n$ if and only if the block matrix

$$\boldsymbol{P}_{(n,m)} := \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{x}_n) & \boldsymbol{M}(\boldsymbol{x}_m) \\ \boldsymbol{M}(\boldsymbol{x}_n)^{\mathrm{T}} & \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{r}_{(n,m)})^{\mathrm{T}} \\ \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} & \boldsymbol{M}(\boldsymbol{r}_{(n,m)}) & \boldsymbol{I}_2 \end{bmatrix} \in \mathbb{R}^{6 \times 6}$$

is positive semi-definite and has rank two.

PROOF OF LEMMA 2 (BEINERT, B., STEIDL[1]).

- Using SCHUR's theorem to neglect the $\boldsymbol{M}(\cdot)$ and translate the positive semidefiniteness.
- The rank property, is, due to the $\boldsymbol{I}_2$, a "lowest"-rank property, such that the SCHUR complement is a full-zero block. □

– Worth mentioning that both problems, the complex- and real-valued (relaxed) problems are equivalent.

- **Notice:** The second components of $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}}\boldsymbol{x}_n$ in the **relaxed real model** originate form the complex-valued model.

- **Notice:** The second components of $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}}\boldsymbol{x}_n$ in the **relaxed real model** originate form the complex-valued model.
- **Claim:** Equivalent real-valued problem

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_1^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m)\in E \tag{$\star$}$$

with $\mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \coloneqq -\sum_{n\in V} w_n \langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle - \sum_{(n,m)\in E} \lambda_{(n,m)}\, \boldsymbol{\ell}_{(n,m)}$.

- **Notice:** The second components of $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}}\boldsymbol{x}_n$ in the **relaxed real model** originate form the complex-valued model.
- **Claim:** Equivalent real-valued problem

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_1^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m) \in E \tag{$\star$}$$

  with $\mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \coloneqq -\sum_{n\in V} w_n \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle - \sum_{(n,m)\in E} \lambda_{(n,m)}\, \boldsymbol{\ell}_{(n,m)}$.
- **Encoding:** The nonconvex constraints $\boldsymbol{x}_n \in \mathbb{S}_1$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle$ for all $(n,m) \in E$:

- **Notice:** The second components of $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}}\boldsymbol{x}_n$ in the **relaxed real model** originate form the complex-valued model.
- **Claim:** Equivalent real-valued problem

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_1^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m)\in E \tag{$\star$}$$

  with $\mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \coloneqq -\sum_{n\in V} w_n \langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle - \sum_{(n,m)\in E} \lambda_{(n,m)}\, \boldsymbol{\ell}_{(n,m)}$.
- **Encoding:** The nonconvex constraints $\boldsymbol{x}_n \in \mathbb{S}_1$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m\rangle$ for all $(n,m)\in E$:

Lemma 3 (Beinert, B., Steidl[1])

Let $n, m \in V$ and $(n,m)\in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_1$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m\rangle$ if and only if

$$\boldsymbol{Q}_{(n,m)} \succeq 0 \;\; \textit{and} \;\; \operatorname{rk}(\boldsymbol{Q}_{(n,m)}) = 2, \;\; \textit{where}\; \boldsymbol{Q}_{(n,m)} \coloneqq \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & 1 & \boldsymbol{\ell}_{(n,m)} \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{\ell}_{(n,m)} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

- **Notice:** The second components of $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}}\boldsymbol{x}_n$ in the **relaxed real model** originate form the complex-valued model.
- **Claim:** Equivalent real-valued problem

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_1^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m)\in E \tag{$\star$}$$

  with $\mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \coloneqq -\sum_{n\in V} w_n \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle - \sum_{(n,m)\in E} \lambda_{(n,m)}\, \boldsymbol{\ell}_{(n,m)}$.
- **Encoding:** The nonconvex constraints $\boldsymbol{x}_n \in \mathbb{S}_1$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle$ for all $(n,m)\in E$:

Lemma 3 (Beinert, B., Steidl[1])

Let $n,m \in V$ and $(n,m)\in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_1$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle$ if and only if

$$\boldsymbol{Q}_{(n,m)} \succeq 0 \ \textit{and}\ \operatorname{rk}(\boldsymbol{Q}_{(n,m)}) = 2, \ \textit{where}\ \boldsymbol{Q}_{(n,m)} \coloneqq \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & 1 & \boldsymbol{\ell}_{(n,m)} \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{\ell}_{(n,m)} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

$$(\star) = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_1^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0 \quad \text{and} \quad \operatorname{rk}(\boldsymbol{Q}_{(n,m)}) = 2.$$

– **Again:** Neglect the rank-two constraint, we propose **our simplified relaxed real model**:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0 \quad \forall (n,m) \in E. \tag{$\star$}$$

- **Again:** Neglect the rank-two constraint, we propose **our simplified relaxed real model**:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0 \quad \forall (n,m) \in E. \tag{$\star$}$$

- Our convex model is simpler w.r.t. to the dimension of the matrix representation than

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{r}\in(\mathbb{R}^2)^M} \mathcal{J}(\boldsymbol{x},\boldsymbol{r}) \quad \text{s.t.} \quad \boldsymbol{P}_{(n,m)} \succeq 0 \quad \forall (n,m) \in E, \tag{$\dagger$}$$

- **Again:** Neglect the rank-two constraint, we propose **our simplified relaxed real model**:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E. \tag{$\star$}$$

- Our convex model is simpler w.r.t. to the dimension of the matrix representation than

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{r}\in(\mathbb{R}^2)^M} \mathcal{J}(\boldsymbol{x},\boldsymbol{r}) \quad \text{s.t.} \quad \boldsymbol{P}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E, \tag{$\dagger$}$$

Theorem 4 (Beinert, B., Steidl[1])

Both models are equivalent in the following sense:

- **Again:** Neglect the rank-two constraint, we propose **our simplified relaxed real model**:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E. \tag{$\star$}$$

- Our convex model is simpler w.r.t. to the dimension of the matrix representation than

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{r}\in(\mathbb{R}^2)^M} \mathcal{J}(\boldsymbol{x},\boldsymbol{r}) \quad \text{s.t.} \quad \boldsymbol{P}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E, \tag{$\dagger$}$$

Theorem 4 (Beinert, B., Steidl[1])

Both models are equivalent in the following sense:

(i) If $(\hat{\boldsymbol{x}},\hat{\boldsymbol{r}})$ solves $(\dagger)$, then $(\hat{\boldsymbol{x}},\Re[\hat{\boldsymbol{r}}])$ solves $(\star)$.

– **Again:** Neglect the rank-two constraint, we propose **our simplified relaxed real model**:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E. \tag{$\star$}$$

– Our convex model is simpler w.r.t. to the dimension of the matrix representation than

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{r}\in(\mathbb{R}^2)^M} \mathcal{J}(\boldsymbol{x},\boldsymbol{r}) \quad \text{s.t.} \quad \boldsymbol{P}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E, \tag{$\dagger$}$$

Theorem 4 (Beinert, B., Steidl[1])

Both models are equivalent in the following sense:

(i) If $(\hat{\boldsymbol{x}},\hat{\boldsymbol{r}})$ solves $(\dagger)$, then $(\hat{\boldsymbol{x}},\Re[\hat{\boldsymbol{r}}])$ solves $(\star)$.

(ii) If $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{\ell}})$ solves $(\star)$, then $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{r}})$ with $\tilde{\boldsymbol{r}}_{(n,m)} = (\tilde{\boldsymbol{\ell}}_{(n,m)}, \Im[\boldsymbol{M}(\tilde{\boldsymbol{x}}_m)^{\mathrm{T}}\tilde{\boldsymbol{x}}_n])^{\mathrm{T}}$ solves $(\dagger)$.

- **Again:** Neglect the rank-two constraint, we propose **our simplified relaxed real model**:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E. \tag{$\star$}$$

- Our convex model is simpler w.r.t. to the dimension of the matrix representation than

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{r}\in(\mathbb{R}^2)^M} \mathcal{J}(\boldsymbol{x},\boldsymbol{r}) \quad \text{s.t.} \quad \boldsymbol{P}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E, \tag{$\dagger$}$$

Theorem 4 (Beinert, B., Steidl[1])

Both models are equivalent in the following sense:

(i) If $(\hat{\boldsymbol{x}},\hat{\boldsymbol{r}})$ solves $(\dagger)$, then $(\hat{\boldsymbol{x}},\Re[\hat{\boldsymbol{r}}])$ solves $(\star)$.

(ii) If $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{\ell}})$ solves $(\star)$, then $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{r}})$ with $\tilde{\boldsymbol{r}}_{(n,m)} = (\tilde{\boldsymbol{\ell}}_{(n,m)}, \Im[\boldsymbol{M}(\tilde{\boldsymbol{x}}_m)^{\mathrm{T}}\tilde{\boldsymbol{x}}_n])^{\mathrm{T}}$ solves $(\dagger)$.

Corollary 5

All *proposed methods are* ***equivalent****.*

- **Main advantage:** the **simplified relaxed real model** is simple to generalize to $(d-1)$-dimensions; let $\mathbb{S}_{d-1} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| = 1\}$.

- **Main advantage:** the **simplified relaxed real model** is simple to generalize to $(d-1)$-dimensions; let $\mathbb{S}_{d-1} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| = 1\}$.
- The **original nonconvex problem** reads as

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{S}_{d-1}^N} \sum_{n \in V} \frac{w_n}{2} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \sum_{(n,m) \in E} \frac{\lambda_{(n,m)}}{2} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2$$

- **Main advantage:** the **simplified relaxed real model** is simple to generalize to $(d-1)$-dimensions; let $\mathbb{S}_{d-1} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| = 1\}$.
- The **original nonconvex problem** reads as

$$\begin{aligned} &\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{S}_{d-1}^N} \sum_{n \in V} \frac{w_n}{2} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \sum_{(n,m) \in E} \frac{\lambda_{(n,m)}}{2} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2 \\ = &\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{S}_{d-1}^N, \boldsymbol{\ell} \in \mathbb{R}^M} \mathcal{K}(\boldsymbol{x}, \boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m) \in E \end{aligned}$$

- **Main advantage:** the **simplified relaxed real model** is simple to generalize to $(d-1)$-dimensions; let $\mathbb{S}_{d-1} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| = 1\}$.
- The **original nonconvex problem** reads as

$$\begin{aligned} &\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_{d-1}^N} \sum_{n\in V} \frac{w_n}{2} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \sum_{(n,m)\in E} \frac{\lambda_{(n,m)}}{2} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2 \\ = &\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_{d-1}^N, \boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m) \in E \end{aligned}$$

Lemma 6 (Beinert, B., Steidl[2])

Let $n, m \in V$ and $(n,m) \in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_{d-1}$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle$ if and only if the block matrix

$$\boldsymbol{Q}_{(n,m)} \coloneqq \begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & 1 & \boldsymbol{\ell}_{(n,m)} \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{\ell}_{(n,m)} & 1 \end{bmatrix} \in \mathbb{R}^{(d+2)\times(d+2)}$$

is positive semi-definite and has rank d.

- **Main advantage:** the **simplified relaxed real model** is simple to generalize to $(d-1)$-dimensions; let $\mathbb{S}_{d-1} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| = 1\}$.
- The **original nonconvex problem** reads as

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{S}_{d-1}^N} \sum_{n \in V} \frac{w_n}{2} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \sum_{(n,m) \in E} \frac{\lambda_{(n,m)}}{2} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2$$

$$= \operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{S}_{d-1}^N, \boldsymbol{\ell} \in \mathbb{R}^M} \mathcal{K}(\boldsymbol{x}, \boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m) \in E$$

Lemma 6 (Beinert, B., Steidl[2])

Let $n, m \in V$ and $(n,m) \in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_{d-1}$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle$ if and only if the block matrix

$$\boldsymbol{Q}_{(n,m)} \coloneqq \begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & 1 & \boldsymbol{\ell}_{(n,m)} \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{\ell}_{(n,m)} & 1 \end{bmatrix} \in \mathbb{R}^{(d+2)\times(d+2)}$$

is positive semi-definite and has rank d.

Convex relaxed Tikhonov problem: $\quad \operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{S}_{d-1}^N, \boldsymbol{\ell} \in \mathbb{R}^M} \mathcal{K}(\boldsymbol{x}, \boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0.$

Non-convex (denoising) problem over manifold

Non-convex (denoising) problem over manifold

↓

Step 1: use manifold constraints to **simplify/rewrite** the objective

General Workflow

Non-convex (denoising) problem over manifold

↓

Step 1: use manifold constraints to **simplify/rewrite** the objective

↓

Step 2: **extend** the set of feasible points to **linearize** the objective

General Workflow

Non-convex (denoising) problem over manifold

↓

Step 1: use manifold constraints to **simplify/rewrite** the objective

↓

Step 2: **extend** the set of feasible points to **linearize** the objective

↓

Step 3: find enrich **matrix representation** for the set of feasible points

Non-convex (denoising) problem over manifold

↓

Step 1: use manifold constraints to **simplify/rewrite** the objective

↓

Step 2: **extend** the set of feasible points to **linearize** the objective

↓

Step 3: find enrich **matrix representation** for the set of feasible points

↓

Step 4: **neclegt** non-convex constraints form the matrix-representation

Non-convex (denoising) problem over manifold

↓

Step 1: use manifold constraints to **simplify/rewrite** the objective

↓

Step 2: **extend** the set of feasible points to **linearize** the objective

↓

Step 3: find enrich **matrix representation** for the set of feasible points

↓

Step 4: **neclegt** non-convex constraints form the matrix-representation

↓

convex-relaxed problem for denoising

– Let us stress to have a common embedding $\mathbb{M} \hookrightarrow \mathbb{R}^{d_{\mathbb{M}}}$ for some dimension $d_{\mathbb{M}} \in \mathbb{N}$

[8] Masahisa Adachi. Vol. 124. Translations of Mathematical Monographs. Providence, RI, USA: AMS, 1993, pp. x +

- Let us stress to have a common embedding $\mathbb{M} \hookrightarrow \mathbb{R}^{d_{\mathbb{M}}}$ for some dimension $d_{\mathbb{M}} \in \mathbb{N}$
- For instance: Any smooth real m-dimensional manifold (by WHITNEY's embedding theorem[8]).

[8] Masahisa Adachi. Vol. 124. Translations of Mathematical Monographs. Providence, RI, USA: AMS, 1993, pp. x +

- Let us stress to have a common embedding $\mathbb{M} \hookrightarrow \mathbb{R}^{d_{\mathbb{M}}}$ for some dimension $d_{\mathbb{M}} \in \mathbb{N}$
- For instance: Any smooth real m-dimensional manifold (by Whitney's embedding theorem[8]).
- We consider noisy data $\boldsymbol{y} = (\boldsymbol{y}_n)_{n\in\mathbb{N}}$ which might be not $\mathbb{M}$-valued. We aim to solve

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N} \frac{1}{2}\sum_{n\in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \frac{\lambda}{2}\sum_{(n,m)\in V} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2 \tag{$\star$}$$

[8] Masahisa Adachi. Vol. 124. Translations of Mathematical Monographs. Providence, RI, USA: AMS, 1993, pp. x +

- Let us stress to have a common embedding $\mathbb{M} \hookrightarrow \mathbb{R}^{d_{\mathbb{M}}}$ for some dimension $d_{\mathbb{M}} \in \mathbb{N}$
- For instance: Any smooth real m-dimensional manifold (by WHITNEY's embedding theorem[8]).
- We consider noisy data $\boldsymbol{y} = (\boldsymbol{y}_n)_{n\in\mathbb{N}}$ which might be not $\mathbb{M}$-valued. We aim to solve

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N} \frac{1}{2}\sum_{n\in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \frac{\lambda}{2}\sum_{(n,m)\in V} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2 \tag{$\star$}$$

- Consider rewriting of $(\star)$ by using

$$\tfrac{1}{2}\|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 = \tfrac{1}{2}\boldsymbol{v}_n - \langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle + \text{const.} \quad \text{if } \|\boldsymbol{x}_n\|^2 = \boldsymbol{v}_n.$$

[8] Masahisa Adachi. Vol. 124. Translations of Mathematical Monographs. Providence, RI, USA: AMS, 1993, pp. x +

- Let us stress to have a common embedding $\mathbb{M} \hookrightarrow \mathbb{R}^{d_{\mathbb{M}}}$ for some dimension $d_{\mathbb{M}} \in \mathbb{N}$
- For instance: Any smooth real m-dimensional manifold (by Whitney's embedding theorem[8]).
- We consider noisy data $\boldsymbol{y} = (\boldsymbol{y}_n)_{n\in\mathbb{N}}$ which might be not $\mathbb{M}$-valued. We aim to solve

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N} \frac{1}{2}\sum_{n\in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \frac{\lambda}{2}\sum_{(n,m)\in V} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2 \qquad (\star)$$

- Consider rewriting of $(\star)$ by using

$$\tfrac{1}{2}\|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 = \tfrac{1}{2}\boldsymbol{v}_n - \langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle + \text{const.} \quad \text{if } \|\boldsymbol{x}_n\|^2 = \boldsymbol{v}_n.$$

- Utilizing the latter, yields

$$(\star) = \operatorname*{arg\,min}_{\substack{\boldsymbol{x}\in\mathbb{M}^N,\\ \boldsymbol{v}\in\mathbb{R}^N,\\ \boldsymbol{\ell}\in\mathbb{R}^\bullet}} \underbrace{\sum_{n\in V}\left(\tfrac{\boldsymbol{v}_n}{2} - \langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle\right) + \lambda \sum_{(n,m)\in V}\left(\tfrac{\boldsymbol{v}_n}{2} + \tfrac{\boldsymbol{v}_m}{2} - \boldsymbol{\ell}_{(n,m)}\right)}_{=:\mathcal{K}(\boldsymbol{x},\boldsymbol{v},\boldsymbol{\ell})} \quad \text{s.t.} \quad \begin{cases} \boldsymbol{v}_n = \|\boldsymbol{x}_n\|^2, \\ \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m\rangle \end{cases}$$

[8] Masahisa Adachi. Vol. 124. Translations of Mathematical Monographs. Providence, RI, USA: AMS, 1993, pp. x +

- Let us stress to have a common embedding $\mathbb{M} \hookrightarrow \mathbb{R}^{d_{\mathbb{M}}}$ for some dimension $d_{\mathbb{M}} \in \mathbb{N}$
- For instance: Any smooth real m-dimensional manifold (by WHITNEY's embedding theorem[8]).
- We consider noisy data $\boldsymbol{y} = (\boldsymbol{y}_n)_{n\in\mathbb{N}}$ which might be not $\mathbb{M}$-valued. We aim to solve

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N} \frac{1}{2}\sum_{n\in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \frac{\lambda}{2}\sum_{(n,m)\in V} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2 \qquad (\star)$$

- Consider rewriting of $(\star)$ by using

$$\tfrac{1}{2}\|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 = \tfrac{1}{2}\boldsymbol{v}_n - \langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle + \text{const.} \quad \text{if } \|\boldsymbol{x}_n\|^2 = \boldsymbol{v}_n.$$

- Utilizing the latter, yields

$$(\star) = \operatorname*{arg\,min}_{\substack{\boldsymbol{x}\in\mathbb{M}^N,\\ \boldsymbol{v}\in\mathbb{R}^N,\\ \boldsymbol{\ell}\in\mathbb{R}^\bullet}} \underbrace{\sum_{n\in V}\left(\tfrac{\boldsymbol{v}_n}{2} - \langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle\right) + \lambda \sum_{(n,m)\in V}\left(\tfrac{\boldsymbol{v}_n}{2} + \tfrac{\boldsymbol{v}_m}{2} - \boldsymbol{\ell}_{(n,m)}\right)}_{=:\mathcal{K}(\boldsymbol{x},\boldsymbol{v},\boldsymbol{\ell})} \quad \text{s.t.} \quad \begin{cases} \boldsymbol{v}_n = \|\boldsymbol{x}_n\|^2, \\ \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m\rangle \end{cases}$$

- Novel matrix representations $\boldsymbol{Q}_{(n,m)}$ yields[1][2][3][4]

$$(\star) = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{R}^{d\times N}, \boldsymbol{v}\in\mathbb{R}^N, \boldsymbol{\ell}\in\mathbb{R}^\bullet} \mathcal{K}(\boldsymbol{x},\boldsymbol{v},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0, \text{min. rk}(\boldsymbol{Q}_{(n,m)}) \quad \forall (n,m)\in E$$

[8] Masahisa Adachi. Vol. 124. Translations of Mathematical Monographs. Providence, RI, USA: AMS, 1993, pp. x +

$\mathbb{M}$	Embedding	$\boldsymbol{Q}_{n,m}, (n,m) \in E$	Remark
$\mathbb{S}_{d-1}$	$\mathbb{R}^d$	$\begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^* & 1 & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \\ \boldsymbol{x}_m^* & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle & 1 \end{bmatrix}$	for $d = 2$ equivalent to (Condat, 2022) via $\begin{bmatrix} 1 & x_n & x_m \\ \bar{x}_n & 1 & \bar{r}_{(n,m)} \\ \bar{x}_m & r_{(n,m)} & 1 \end{bmatrix} \in \mathbb{C}^{3\times 3} \simeq \mathbb{R}^{6\times 6}$
$\mathbb{H}_d$	$\mathbb{R}^{d+1}$	$\begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \tilde{\boldsymbol{x}}_n & \boldsymbol{x}_m & \tilde{\boldsymbol{x}}_m \\ \boldsymbol{x}_n^* & \Vert\boldsymbol{x}_n\Vert^2 & -1 & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle & \eta(\boldsymbol{x}_n, \boldsymbol{x}_m) \\ \tilde{\boldsymbol{x}}_n^* & -1 & \Vert\boldsymbol{x}_n\Vert^2 & \eta(\boldsymbol{x}_n, \boldsymbol{x}_m) & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \\ \boldsymbol{x}_m^* & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle & \eta(\boldsymbol{x}_n, \boldsymbol{x}_m) & \Vert\boldsymbol{x}_m\Vert^2 & -1 \\ \tilde{\boldsymbol{x}}_m^* & \eta(\boldsymbol{x}_n, \boldsymbol{x}_m) & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle & -1 & \Vert\boldsymbol{x}_m\Vert^2 \end{bmatrix}$	$\tilde{\boldsymbol{\xi}} \coloneqq (\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_d, -\boldsymbol{\eta}_{d+1}) \in \mathbb{R}^{d+1}$
$\mathbb{S}_0^d$	$\mathbb{R}^d$	$\begin{bmatrix} 1 & \boldsymbol{x}_n^* & \boldsymbol{x}_m^* \\ \boldsymbol{x}_n & \boldsymbol{I}_d + (x_{n,i} x_{n,j})_{i\neq j} & \boldsymbol{x}_n \boldsymbol{x}_m^* \\ \boldsymbol{x}_m & \boldsymbol{x}_n \boldsymbol{x}_m^* & \boldsymbol{I}_d + (x_{m,i} x_{m,j})_{i\neq j} \end{bmatrix}$	
$\mathbb{T}_d$	$\mathbb{R}^{2\times d}$	$\begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^* & \boldsymbol{I}_d + (\langle \boldsymbol{x}_{n,i}, \boldsymbol{x}_{n,j} \rangle)_{i\neq j} & \boldsymbol{x}_n^* \boldsymbol{x}_m \\ \boldsymbol{x}_m^* & \boldsymbol{x}_m^* \boldsymbol{x}_n & \boldsymbol{I}_d + (\langle \boldsymbol{x}_{m,i}, \boldsymbol{x}_{m,j} \rangle)_{i\neq j} \end{bmatrix}$	$\mathbb{T}_d \simeq \times_{i=1}^d \mathbb{S}_1$
$\mathbb{V}_d(k)$	$\mathbb{R}^{d\times k}$	$\begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^* & \boldsymbol{I}_k & \boldsymbol{x}_n^* \boldsymbol{x}_m \\ \boldsymbol{x}_m^* & \boldsymbol{x}_m^* \boldsymbol{x}_n & \boldsymbol{I}_k \end{bmatrix}$	$\mathbb{V}_d(1) \simeq \mathbb{S}_{d-1}$
SO(3)	$Q(8) \hookrightarrow \mathbb{S}_3 \simeq \mathbb{S}_{\mathbb{H}}$		other dimensions are employable.
SE(2)	$\mathbb{S}_1 \times \mathbb{R}^2 \subset \mathbb{R}^4$	$\begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^* & \boldsymbol{I}_k & \boldsymbol{x}_n^* \boldsymbol{x}_m \\ \boldsymbol{x}_m^* & \boldsymbol{x}_m^* \boldsymbol{x}_n & \boldsymbol{I}_k \end{bmatrix}$	other dimensions are employable.

Table of Contents

From Tikhonov to TV

– The Tikhonov regularization (as in the latter section) is suitable for smooth signals.

- The Tikhonov regularization (as in the latter section) is suitable for smooth signals.
- We want to **recover piecewise constant signals** from noisy measurements.

- The Tikhonov regularization (as in the latter section) is suitable for smooth signals.
- We want to **recover piecewise constant signals** from noisy measurements.
- For this reason, we **replace the squared Euclidean norm** of the regularizer **by the Manhattan norm**.

- The TIKHONOV regularization (as in the latter section) is suitable for smooth signals.
- We want to **recover piecewise constant signals** from noisy measurements.
- For this reason, we **replace the squared EUCLIDEAN norm** of the regularizer **by the Manhattan norm**.
- Yielding an anisotropic TOTAL VARIATION (TV) regularization:

- The Tikhonov regularization (as in the latter section) is suitable for smooth signals.
- We want to **recover piecewise constant signals** from noisy measurements.
- For this reason, we **replace the squared Euclidean norm** of the regularizer **by the Manhattan norm**.
- Yielding an anisotropic Total Variation (TV) regularization:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N} \frac{1}{2}\sum_{n\in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|_2^2 + \mu\mathrm{TV}(\boldsymbol{x}) \quad \text{with} \quad \mathrm{TV}(\boldsymbol{x}) \coloneqq \sum_{(n,m)\in E} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|_1. \tag{$\star$}$$

- The Tikhonov regularization (as in the latter section) is suitable for smooth signals.
- We want to **recover piecewise constant signals** from noisy measurements.
- For this reason, we **replace the squared Euclidean norm** of the regularizer **by the Manhattan norm**.
- Yielding an anisotropic Total Variation (TV) regularization:

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{M}^N} \frac{1}{2} \sum_{n \in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|_2^2 + \mu \mathrm{TV}(\boldsymbol{x}) \quad \text{with} \quad \mathrm{TV}(\boldsymbol{x}) \coloneqq \sum_{(n,m) \in E} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|_1. \tag{$\star$}$$

- Using the property of the squared 2-norm again, we obtain

$$\mathcal{K}(\boldsymbol{x}, \boldsymbol{v}) \coloneqq \sum_{n \in V} \left(\tfrac{\boldsymbol{v}_n}{2} - \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle \right) + \mu \mathrm{TV}(\boldsymbol{x}) \quad \text{with} \quad (\star) = \operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{M}^N, \boldsymbol{v} \in \mathbb{R}^N} \mathcal{K}(\boldsymbol{x}, \boldsymbol{v}) \quad \text{s.t} \quad \boldsymbol{v}_n = \|\boldsymbol{x}_n\|^2.$$

- The TIKHONOV regularization (as in the latter section) is suitable for smooth signals.
- We want to **recover piecewise constant signals** from noisy measurements.
- For this reason, we **replace the squared EUCLIDEAN norm** of the regularizer **by the Manhattan norm**.
- Yielding an anisotropic TOTAL VARIATION (TV) regularization:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N} \frac{1}{2}\sum_{n\in V}\|\boldsymbol{x}_n-\boldsymbol{y}_n\|_2^2+\mu\mathrm{TV}(\boldsymbol{x}) \quad \text{with} \quad \mathrm{TV}(\boldsymbol{x}) \coloneqq \sum_{(n,m)\in E}\|\boldsymbol{x}_n-\boldsymbol{x}_m\|_1. \tag{$\star$}$$

- Using the property of the squared 2-norm again, we obtain

$$\mathcal{K}(\boldsymbol{x},\boldsymbol{v}) \coloneqq \sum_{n\in V}\left(\tfrac{\boldsymbol{v}_n}{2}-\langle\boldsymbol{x}_n,\boldsymbol{y}_n\rangle\right)+\mu\mathrm{TV}(\boldsymbol{x}) \quad \text{with} \quad (\star)=\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N,\boldsymbol{v}\in\mathbb{R}^N}\mathcal{K}(\boldsymbol{x},\boldsymbol{v}) \quad \text{s.t} \quad \boldsymbol{v}_n=\|\boldsymbol{x}_n\|^2.$$

PROPOSITION 1

Let $\mathbb{M}$ be one of the considered manifolds, $d_{\mathbb{M}}\in\mathbb{N}$ the dimension of the used embedding in the Euclidean vector space and $\boldsymbol{x}_n\in\mathbb{R}^{d_{\mathbb{M}}}$. Then, $\boldsymbol{x}_n,\boldsymbol{x}_m\in\mathbb{M}$ for $(n,m)\in E$ iff $\boldsymbol{V}_n$ or $\boldsymbol{Q}_{(n,m)}\succeq 0$ with minimal rank.

- The TIKHONOV regularization (as in the latter section) is suitable for smooth signals.
- We want to **recover piecewise constant signals** from noisy measurements.
- For this reason, we **replace the squared EUCLIDEAN norm** of the regularizer **by the Manhattan norm**.
- Yielding an anisotropic TOTAL VARIATION (TV) regularization:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N} \frac{1}{2}\sum_{n\in V}\|\boldsymbol{x}_n-\boldsymbol{y}_n\|_2^2+\mu\mathrm{TV}(\boldsymbol{x}) \quad \text{with} \quad \mathrm{TV}(\boldsymbol{x}) \coloneqq \sum_{(n,m)\in E}\|\boldsymbol{x}_n-\boldsymbol{x}_m\|_1. \tag{$\star$}$$

- Using the property of the squared 2-norm again, we obtain

$$\mathcal{K}(\boldsymbol{x},\boldsymbol{v}) \coloneqq \sum_{n\in V}\left(\tfrac{\boldsymbol{v}_n}{2}-\langle\boldsymbol{x}_n,\boldsymbol{y}_n\rangle\right)+\mu\mathrm{TV}(\boldsymbol{x}) \quad \text{with} \quad (\star)=\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N,\boldsymbol{v}\in\mathbb{R}^N}\mathcal{K}(\boldsymbol{x},\boldsymbol{v}) \quad \text{s.t} \quad \boldsymbol{v}_n=\|\boldsymbol{x}_n\|^2.$$

PROPOSITION 1

Let $\mathbb{M}$ be one of the considered manifolds, $d_{\mathbb{M}}\in\mathbb{N}$ the dimension of the used embedding in the Euclidean vector space and $\boldsymbol{x}_n\in\mathbb{R}^{d_{\mathbb{M}}}$. Then, $\boldsymbol{x}_n,\boldsymbol{x}_m\in\mathbb{M}$ for $(n,m)\in E$ iff $\boldsymbol{V}_n$ or $\boldsymbol{Q}_{(n,m)}\succeq 0$ with minimal rank.

- Convexifying the $\mathbb{M}$-valued domain, we propose **our relaxed, convex problem**:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{R}^{d_{\mathbb{M}}\times N},\boldsymbol{v}\in\mathbb{R}^N}\mathcal{K}(\boldsymbol{x},\boldsymbol{v}) \quad \text{s.t.} \quad \boldsymbol{V}_n\succeq 0 \quad \text{for all} \quad n\in V. \tag{$\dagger$}$$

- If $\mathbb{M} = \mathbb{S}_0^d$ the relaxed problem ($\dagger$) becomes a tight convexification of the nonconvex problem ($\star$).

- If $\mathbb{M} = \mathbb{S}_0^d$ the relaxed problem $(\dagger)$ becomes a tight convexification of the nonconvex problem $(\star)$.
- Consider first the case $d = 1$[2]; for arbitrary $d \in \mathbb{N}$ the same can be proven[4] (coordinate-wise).

- If $\mathbb{M} = \mathbb{S}_0^d$ the relaxed problem $(\dagger)$ becomes a tight convexification of the nonconvex problem $(\star)$.
- Consider first the case $d = 1$[2]; for arbitrary $d \in \mathbb{N}$ the same can be proven[4] (coordinate-wise).
- We introduce the characteristic χ_η regarding the level $\eta \in [-1, 1] =: \mathbb{B}_1$ of a signal $\boldsymbol{x} := (\boldsymbol{x}_n)_{n \in V} \in \mathbb{B}_1^N$

- If $\mathbb{M} = \mathbb{S}_0^d$ the relaxed problem (†) becomes a tight convexification of the nonconvex problem ($\star$).
- Consider first the case $d = 1$[2]; for arbitrary $d \in \mathbb{N}$ the same can be proven[4] (coordinate-wise).
- We introduce the characteristic χ_η regarding the level $\eta \in [-1, 1] =: \mathbb{B}_1$ of a signal $\boldsymbol{x} := (\boldsymbol{x}_n)_{n \in V} \in \mathbb{B}_1^N$

$$\chi_\eta(\boldsymbol{x}) := (\chi_\eta(\boldsymbol{x}_n))_{n \in V} \qquad \text{with} \qquad \chi_\eta(\boldsymbol{x}_n) := \begin{cases} 1 & \text{if } \boldsymbol{x}_n > \eta, \\ -1 & \text{if } \boldsymbol{x}_n \leq \eta. \end{cases}$$

- If $\mathbb{M} = \mathbb{S}_0^d$ the relaxed problem (†) becomes a tight convexification of the nonconvex problem ($\star$).
- Consider first the case $d = 1$[2]; for arbitrary $d \in \mathbb{N}$ the same can be proven[4] (coordinate-wise).
- We introduce the characteristic χ_η regarding the level $\eta \in [-1, 1] =: \mathbb{B}_1$ of a signal $\boldsymbol{x} := (\boldsymbol{x}_n)_{n \in V} \in \mathbb{B}_1^N$

$$\chi_\eta(\boldsymbol{x}) := (\chi_\eta(\boldsymbol{x}_n))_{n \in V} \qquad \text{with} \qquad \chi_\eta(\boldsymbol{x}_n) := \begin{cases} 1 & \text{if } \boldsymbol{x}_n > \eta, \\ -1 & \text{if } \boldsymbol{x}_n \leq \eta. \end{cases}$$

- **Remark:** For any $\boldsymbol{x} := (\boldsymbol{x}_n)_{n \in V} \in \mathbb{B}_1^N$, the characteristic $\chi_\eta(\boldsymbol{x})$ is a binary signal.

- If $\mathbb{M} = \mathbb{S}_0^d$ the relaxed problem (†) becomes a tight convexification of the nonconvex problem ($\star$).
- Consider first the case $d = 1$[2]; for arbitrary $d \in \mathbb{N}$ the same can be proven[4] (coordinate-wise).
- We introduce the characteristic χ_η regarding the level $\eta \in [-1, 1] =: \mathbb{B}_1$ of a signal $\boldsymbol{x} := (\boldsymbol{x}_n)_{n \in V} \in \mathbb{B}_1^N$

$$\chi_\eta(\boldsymbol{x}) := (\chi_\eta(\boldsymbol{x}_n))_{n \in V} \qquad \text{with} \qquad \chi_\eta(\boldsymbol{x}_n) := \begin{cases} 1 & \text{if } \boldsymbol{x}_n > \eta, \\ -1 & \text{if } \boldsymbol{x}_n \leq \eta. \end{cases}$$

- **Remark:** For any $\boldsymbol{x} := (\boldsymbol{x}_n)_{n \in V} \in \mathbb{B}_1^N$, the characteristic $\chi_\eta(\boldsymbol{x})$ is a binary signal.

Lemma 7 (Beinert, B.[2])

For $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{B}_1$, *it holds* $|\boldsymbol{x}_n - \boldsymbol{x}_m| = \frac{1}{2} \int_{-1}^{1} |\chi_\eta(\boldsymbol{x}_n) - \chi_\eta(\boldsymbol{x}_m)| \, \mathrm{d}\eta.$

- If $\mathbb{M} = \mathbb{S}_0^d$ the relaxed problem $(\dagger)$ becomes a tight convexification of the nonconvex problem $(\star)$.
- Consider first the case $d = 1$[2]; for arbitrary $d \in \mathbb{N}$ the same can be proven[4] (coordinate-wise).
- We introduce the characteristic χ_η regarding the level $\eta \in [-1, 1] =: \mathbb{B}_1$ of a signal $\boldsymbol{x} := (\boldsymbol{x}_n)_{n \in V} \in \mathbb{B}_1^N$

$$\chi_\eta(\boldsymbol{x}) := (\chi_\eta(\boldsymbol{x}_n))_{n \in V} \qquad \text{with} \qquad \chi_\eta(\boldsymbol{x}_n) := \begin{cases} 1 & \text{if } \boldsymbol{x}_n > \eta, \\ -1 & \text{if } \boldsymbol{x}_n \leq \eta. \end{cases}$$

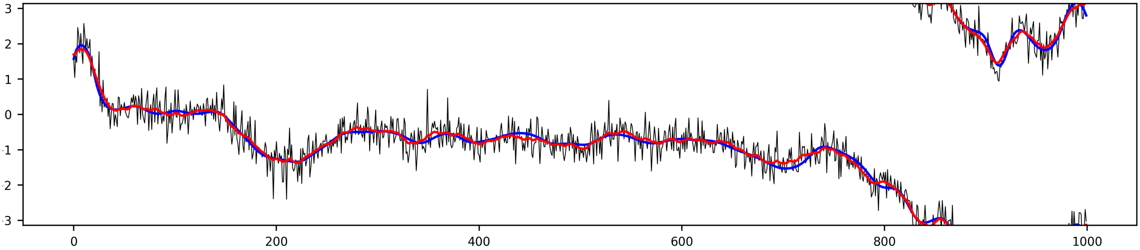
- **Remark:** For any $\boldsymbol{x} := (\boldsymbol{x}_n)_{n \in V} \in \mathbb{B}_1^N$, the characteristic $\chi_\eta(\boldsymbol{x})$ is a binary signal.

Lemma 7 (Beinert, B.[2])

For $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{B}_1$, it holds $|\boldsymbol{x}_n - \boldsymbol{x}_m| = \frac{1}{2} \int_{-1}^{1} |\chi_\eta(\boldsymbol{x}_n) - \chi_\eta(\boldsymbol{x}_m)| \, \mathrm{d}\eta$.

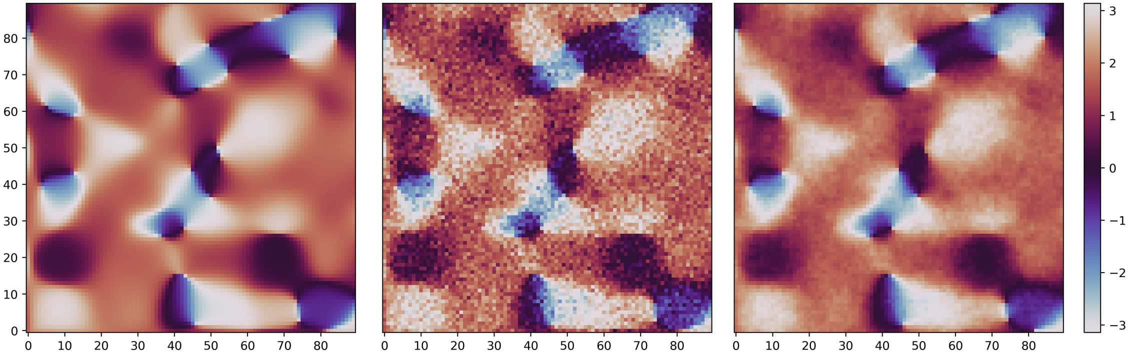
Theorem 8 (Beinert, B.[2])

Let $\tilde{\boldsymbol{x}} \in \mathbb{B}_1^N$ be a solution of $(\dagger)$ for $d = 1$. Then, $\chi_\eta(\tilde{\boldsymbol{x}})$ is a solution of $(\star)$ for almost all $\eta \in [-1, 1]$.



- synthetic, smooth signal $(\boldsymbol{x}_n)_{n\in V}$ on the line graph.
- noisy observation $(\boldsymbol{y}_n)_{n\in V}$ are generated using the VON MISES–FISHER distribution by $\boldsymbol{y}_n \sim \mathcal{N}_{\mathrm{vMF}}(\boldsymbol{x}_n, \kappa)$ for all $n \in V$, where $\kappa > 0$ is the capacity, here $\kappa = 10$.

	mean	time	iter.	parameters	reg. parameters
PMM on **relaxed complex**	10^{-9}	2.10	201	$\tau = 0.1, \sigma = (4\tau)^{-1}$	
ADMM on **relaxed real**	10^{-12}	1.83	182	$\rho = 3$	$w_n = 1, \lambda_{(n,m)} = 25$
ADMM on **simp. relaxed real**	10^{-13}	1.77	181	$\rho = 3$	



	mean	time	iter.	parameters	reg. parameters
PMM on **relaxed complex**	10^{-4}	747.18	5536	$\tau = 0.1, \sigma = (8\tau)^{-1}$	
ADMM on **relaxed real**	10^{-4}	389.01	2457	$\rho = 20$	$w_n = 1, \lambda_{(n,m)} = 1$
ADMM on **simp. relaxed real**	10^{-4}	301.08	1943	$\rho = 20$	

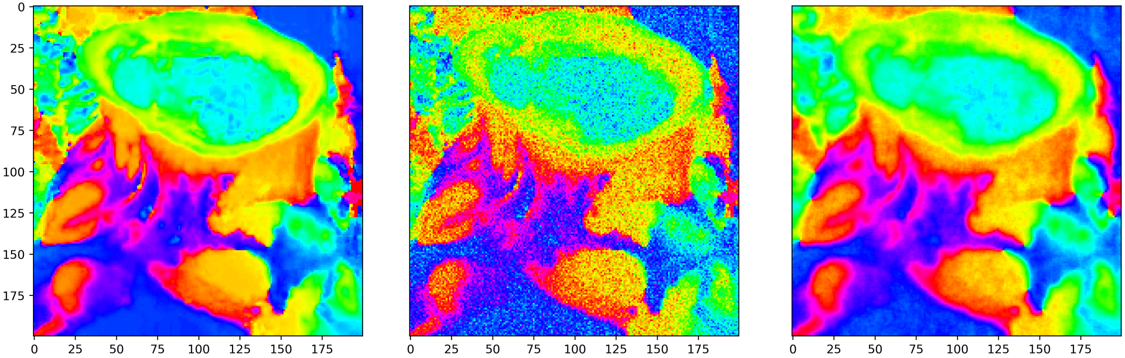


FIGURE 2: Hue experiment: Ground truth (left), noisy observation with $\kappa = 10$ (middle), and denoised version (right) ADMM has been employed with $w_n := 1$, $\lambda_{(n,m)} := 2$, and $\rho := 20$.

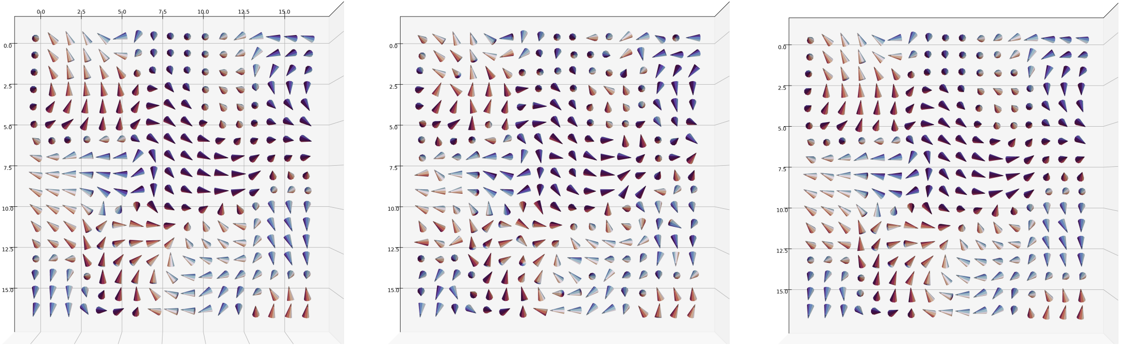


Figure 3: SO(3)-valued image graph: Ground truth (left), noisy observation with $\kappa_1 := 30$ and $\kappa_2 := 5$ (middle), and denoised version (right) The rotations are visualized by operating on a colored 3d cone. The entire signal consists of 90×90 pixel, where only every third pixel from the 25th to the 75th pixel is visualized. The parameters have been $w_n := 1$, $\lambda_{(n,m)} := 1$, and $\rho := 3$.

Table of Contents

Figure 4: Averages for 50 randomly generated QR codes for different noise levels.

$\sigma =$	Algorithm	signal errors MSE	signal errors MIoU	λ	time (sec.)	distance to sphere
$\sqrt{2}\frac{5}{10}$	fast-TV[a]	**0.00036**	**0.99734**	0.7	< 0.1	0.07879
	ANISO-TV[b]	0.00112	0.97495	1.9	1.8	0.00122
	ADMM-TV	**0.00036**	**0.99734**	0.7	1.1	**0.00000**
$\sqrt{2}\frac{7}{10}$	fast-TV	0.00069	0.99018	1.0	< 0.1	0.10021
	ANISO-TV	0.00190	0.92942	2.4	1.9	0.00732
	ADMM-TV	**0.00069**	**0.99030**	1.0	1.1	**0.00000**
$\sqrt{2}\frac{9}{10}$	fast-TV	0.00106	0.97741	1.7	< 0.1	0.12263
	ANISO-TV	0.00263	0.86956	2.9	2.1	0.01445
	ADMM-TV	**0.00106**	**0.97753**	1.7	1.2	**0.00000**

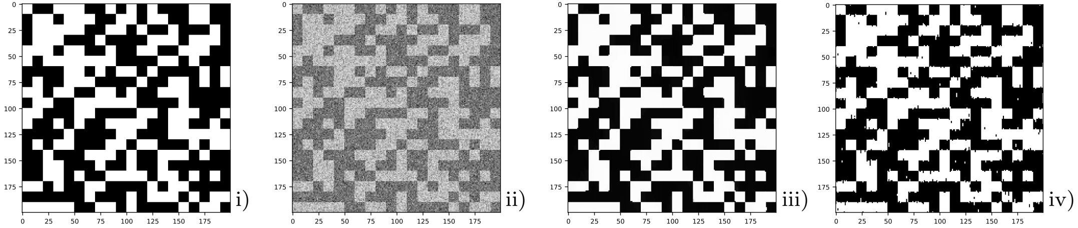
[a] Laurent Condat. 2012. URL: https://hal.science/hal-00675043, Laurent Condat. In: *IEEE Signal Process. Lett.* 20.11 (2013), pp. 1054–1057.

[b] Rustum Choksi, Yves van Gennip, and Adam Oberman. arXiv:1007.1035. 2017.

Figure 5: Averages for 20 randomly generated noisy instances of the ground truth for different noise levels.

$\kappa =$	Algorithm	signal error MSE	λ	time (sec.)	distance to sphere
50	CPPA-TV[c]	0.00076	0.15	128.2	—
	fast-TV[a]	0.00056	0.25	< 0.1	0.00055
	ADMM-TV	**0.00043**	0.2	4.3	**0.00000**
20	CPPA-TV	0.00198	0.25	166.1	—
	fast-TV	0.00199	0.25	< 0.1	0.00098
	ADMM-TV	**0.00107**	0.3	6.2	**0.00000**
10	CPPA-TV	0.00409	0.55	191.7	—
	fast-TV	0.00388	0.25	< 0.1	0.00155
	ADMM-TV	**0.00218**	0.45	10.5	**0.00000**

[c] R. Bergmann et al. In: *SIAM J. Imaging Sci.* 7.4 (2014), pp. 2916–2953



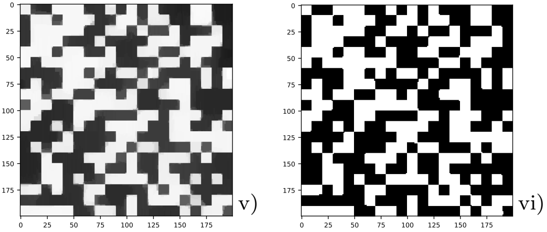


Figure 6: QR code denoising example: (i) ground truth, (ii) noisy data ($\sigma = \sqrt{2} \cdot 0.5$), (iii) ADMM-TV ($\lambda = 1.0$, $\rho = 0.1$) without projection, (iv) ANISO-TV ($\lambda = 1.6$) with projection χ_0, (v) fast-TV ($\lambda = 1.0$) without projection, (vi) fast-TV ($\lambda = 1.0$) with projection χ_0.

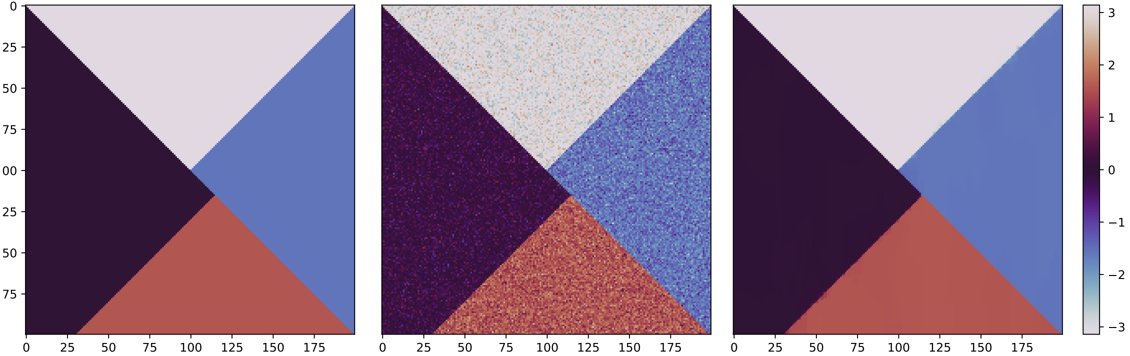


Figure 7: Toy-data example for $\mathbb{S}_1$-image denoising (from left to right): (i) ground truth, (ii) noisy measurement generated by the von Mises–Fisher distribution with $\kappa = 10$, (iii) solution via ADMM-TV ($\lambda = 0.55$, $\rho = 10$) without final projection.

[8] Example following Robin Kenis, Emanuel Laude, and Panagiotis Patrinos. arXiv:2308.00079. 2023

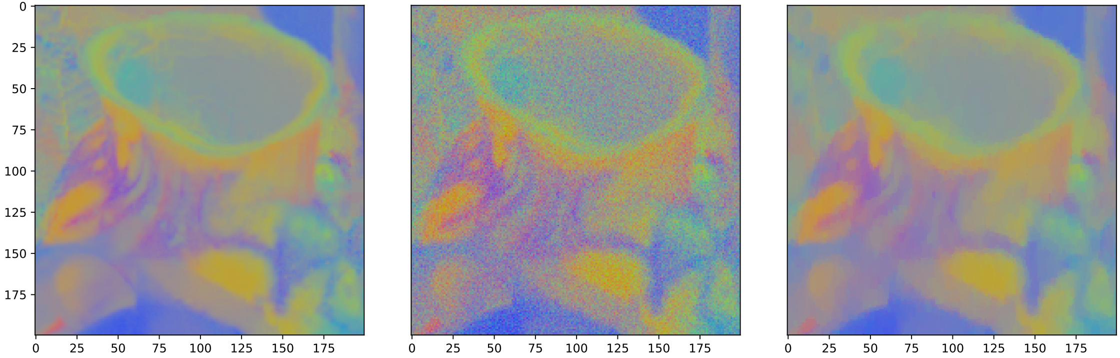


FIGURE 8: Chromaticity-denoising for the coral image from in the RGB color space (from left to right): (i) ground truth, (ii) noisy measurement generated by the VON MISES–FISHER distribution with capacity $\kappa = 200$, (iii) solution via proposed model without final projection and with MSE $9.46 \cdot 10^{-4}$ and averaged distance to the sphere $1.34 \cdot 10^{-6}$.

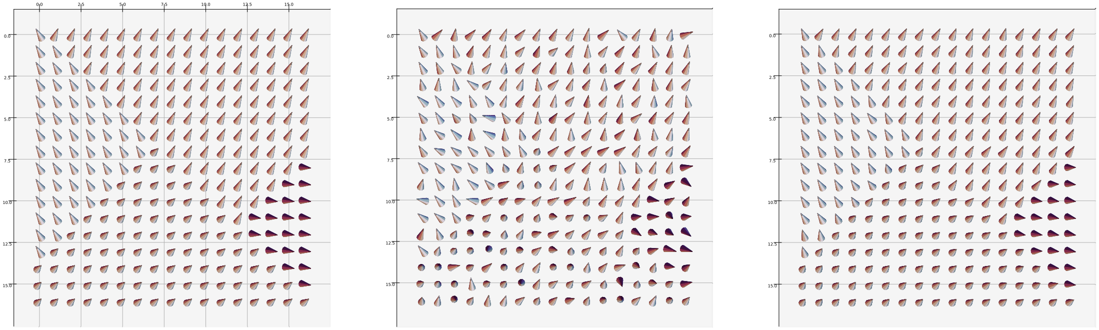


Figure 9: Denoising of a synthetic SO(3)-image (90×90 pixels) (from left to right, downsampled results): (i) ground truth, (ii) noisy measurement generated by the von Mises–Fisher distribution with capacity $\kappa_1 = 10$ for the rotation angles and $\kappa_2 = 10$ for the rotation axis. (iii) solution proposed model without final projection and with MSE $5.889 \cdot 10^{-3}$ and averaged distance to the unit quaternions $1.45 \cdot 10^{-7}$.

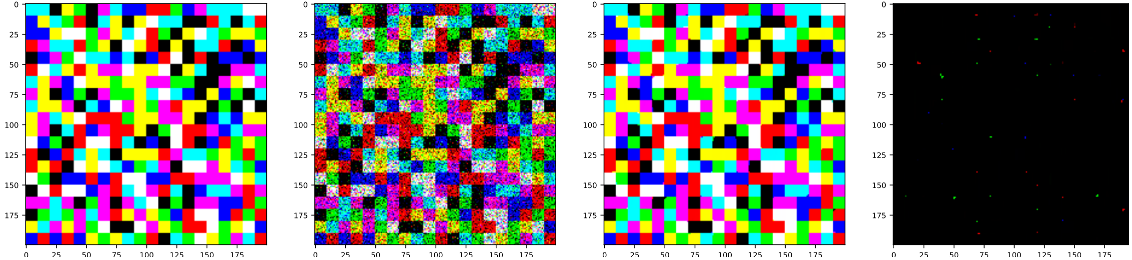


FIGURE 10: Multi-color QR code denoising from left to right): ground truth (first), corrupted image by (3-dimensional) GAUSSIAN noise with standard deviation $\sqrt{2} \cdot 0.5$ (second), restoration and the restoration error (forth). The mean absolute distance of the restored pixels to $\mathbb{S}_0^3$ is in order of 10^{-5}, i.e., here a solution of the non-convex TV model without the proven projection procedure.

Table of Contents

– Series of K noisy instances, assume each pixel $x_{i,j}^{(k)}$ follows a Gaussian distribution.

[9] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- Series of K noisy instances, assume each pixel $x_{i,j}^{(k)}$ follows a Gaussian distribution.
- Empirical mean and variance may be estimated

$$\hat{\mu}_{ij} := \frac{1}{K}\sum_{k=1}^{K} x_{i,j}^{(k)}, \qquad \hat{\sigma}_{ij}^2 := \frac{1}{K}\sum_{k=1}^{K}(x_{i,j}^{(k)} - \hat{\mu}_{ij})^2. \tag{1}$$

[9] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- Series of K noisy instances, assume each pixel $x_{i,j}^{(k)}$ follows a GAUSSIAN distribution.
- Empirical mean and variance may be estimated

$$\hat{\mu}_{ij} := \frac{1}{K}\sum_{k=1}^{K} x_{i,j}^{(k)}, \qquad \hat{\sigma}_{ij}^2 := \frac{1}{K}\sum_{k=1}^{K}(x_{i,j}^{(k)} - \hat{\mu}_{ij})^2. \tag{1}$$

- Set of GAUSSIANS $\mathcal{N}(\mu, \sigma)$ parameterized by $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_{>0}$ and equipped with the so-called FISHER metric is isometric to the hyperbolic space $\mathbb{H}_2$.

[9] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

- Series of K noisy instances, assume each pixel $x_{i,j}^{(k)}$ follows a GAUSSIAN distribution.
- Empirical mean and variance may be estimated

$$\hat{\mu}_{ij} := \frac{1}{K}\sum_{k=1}^{K} x_{i,j}^{(k)}, \qquad \hat{\sigma}_{ij}^2 := \frac{1}{K}\sum_{k=1}^{K}(x_{i,j}^{(k)} - \hat{\mu}_{ij})^2. \tag{1}$$

- Set of GAUSSIANS $\mathcal{N}(\mu,\sigma)$ parameterized by $(\mu,\sigma) \in \mathbb{R}\times\mathbb{R}_{>0}$ and equipped with the so-called FISHER metric is isometric to the hyperbolic space $\mathbb{H}_2$.
- The isometry is given by $\pi_3 \circ \pi_2 \circ \pi_1 \colon \mathcal{N} \to \mathbb{H}_2 := \{\boldsymbol{\xi} \in \mathbb{R}^3 : \boldsymbol{\xi}_1^2 + \boldsymbol{\xi}_2^2 - \boldsymbol{\xi}_3^2 = -1, \boldsymbol{\xi}_3 > 0\}$ with[9]

$$\pi_1 \colon \mathcal{N} \to \mathbb{P}_2, \qquad (\mu,\sigma) \mapsto \tfrac{1}{\sqrt{2}}(\mu, \sqrt{2}\sigma)^{\mathrm{T}}, \tag{2a}$$

$$\pi_2 \colon \mathbb{P}_2 \to \mathbb{D}_2, \qquad \boldsymbol{y} \mapsto \tfrac{1}{y_1^2+(y_2+1)^2}(2y_1, \|\boldsymbol{y}\|_2^2 - 1)^{\mathrm{T}}, \tag{2b}$$

$$\pi_3 \colon \mathbb{D}_2 \to \mathbb{H}_2, \qquad \boldsymbol{y} \mapsto \tfrac{1}{1-\|\boldsymbol{y}\|_2^2}(2y_1, 2y_2, 1 + \|\boldsymbol{y}\|_2^2)^{\mathrm{T}}. \tag{2c}$$

[9] R. Beinert and J. Bresch. arXiv:2506.22826. 2025.

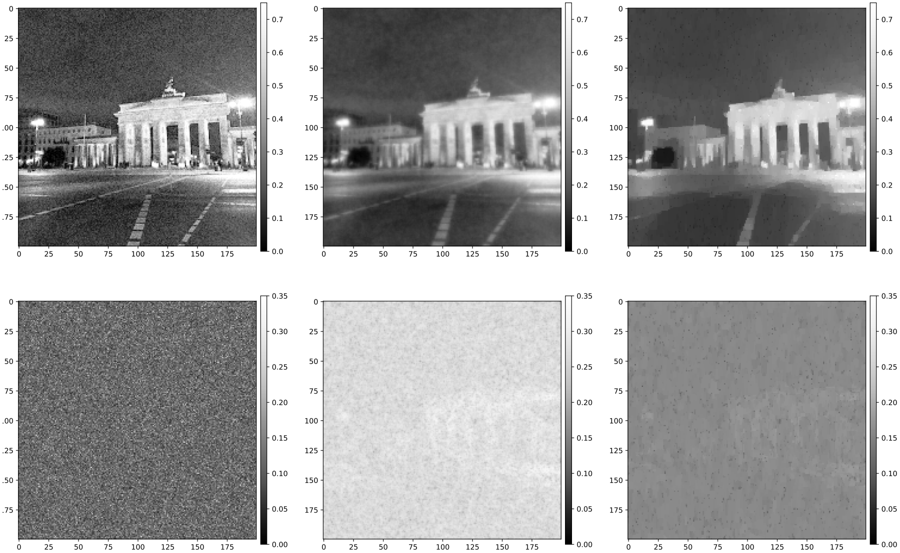


Figure 11: Empirical mean (top) and standard deviation (bottom) for the Brandenburg Gate image. Here, $K = 20$ shots are randomly sampled by adding white noise with $\sigma = 0.15$. The empirical quantities are shown on the left, the Tikhonov denoised images in the middle, and the TV denoised images on the right.

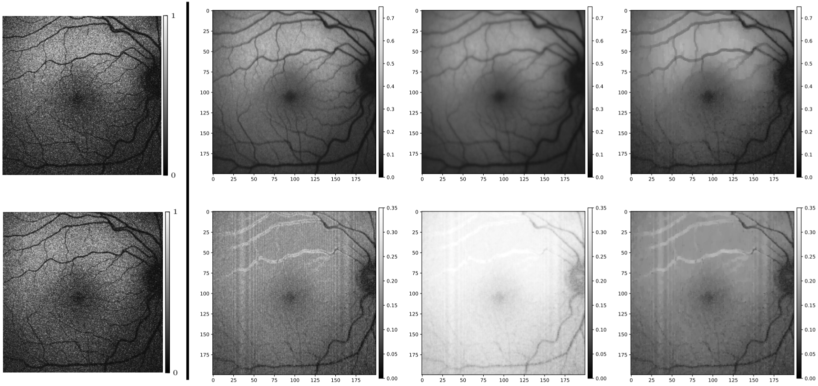


Figure 12: Clearing the estimated mean (top) and standard deviation (bottom) of $K = 20$ retina scans. *Leftmost:* the 1st (top) and 20th (bottom) image of the retina scans. Tikhonov (left) TV denoised (right).

TV-model						PDRA[10]			
SNR($\hat{\mu}$)	SNR($\hat{\sigma}$)	μ	$\mathcal{K}$	MAE $\eta \equiv -1$	time (sec.)	SNR($\hat{\mu}$)	SNR($\hat{\sigma}$)	α	time (sec.)
6.121	12.211	0.05	$2.61 \cdot 10^5$	$4.73 \cdot 10^{-4}$	160	5.963	12.400	0.1	120
6.001	12.369	0.10	$2.54 \cdot 10^5$	$4.78 \cdot 10^{-4}$	180	5.911	12.618	0.3	140
5.986	**12.301**	0.15	$1.81 \cdot 10^5$	$4.47 \cdot 10^{-4}$	190	5.874	12.748	0.5	280
5.966	12.455	0.20	$1.19 \cdot 10^5$	$4.67 \cdot 10^{-4}$	220	5.843	12.849	0.7	430

[10] R. Bergmann, J. Persch, and G. Steidl. In: *SIAM J. Imaging Sci.* 9.3 (2016), pp. 901–937.

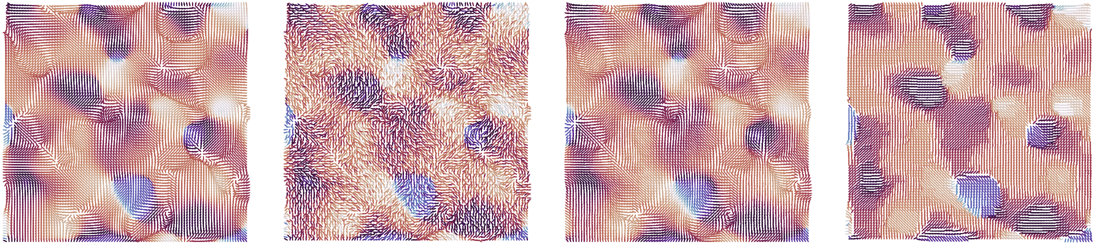


Figure 13: Comparison of both ansätze: Tikhonov-like (second from the right) and TV-like (first from the right) for $\mathbb{T}_2 \simeq \mathbb{S}_1 \times \mathbb{S}_1$-valued images. Ground-truth (first from the left) and noisy image (second from the left) by von Mises–Fischer-distribution ($\kappa = 10$). The plateau effect due to the isotropic TV can be observed. Visualized by the periodic angles: first the arrow, second the colour.

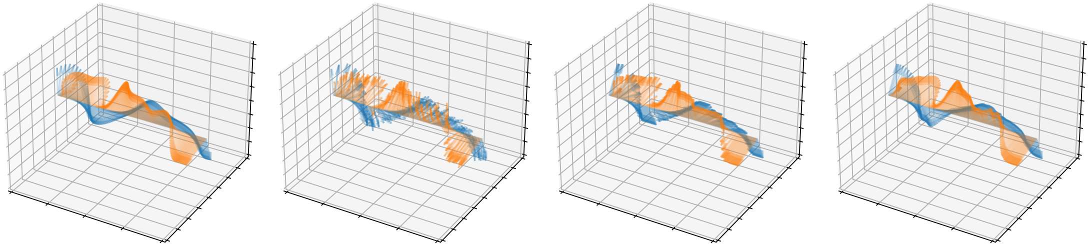


FIGURE 14: Restoration of a $\mathbb{V}_3(2)$-valued signal (from left to tight): ground truth (left), noisy measurement (middle left), and restored signal (convexified TV model and convexified TIKHONOV model). In both cases the mean absolute distance of the norm of the recovered basis vectors to one is of order 10^{-5}, the mean absolute distance of the inner product between the recovered basis vectors to zero is of order 10^{-4}.

Thank you for your attention!

[1] Masahisa Adachi. *Embeddings and Immersions*. Vol. 124. Translations of Mathematical Monographs. Providence, RI, USA: AMS, 1993, pp. x + 183. DOI: 10.1090/mmono/124.

[2] R. Beinert and J. Bresch. "Denoising hyperbolic-valued data by relaxed regularizations". In: *International Conference on Scale Space and Variational Methods in Computer Vision (SSVM)*. Vol. 15667. Lect. Notes Comput. Sci. 2. Dartington, UK: Springer, 2025, pp. 16–29. DOI: 10.1007/978-3-031-92366-1_2.

[3] R. Beinert and J. Bresch. *Denoising multi-color QR codes and Stiefel-valued data by relaxed regularizations*. arXiv:2506.22826. 2025. DOI: 10.48550/arXiv.2506.22826.

[4] R. Beinert and J. Bresch. "Denoising sphere-valued data by relaxed total variation regularization". In: *Proc. Appl. Math. Mech.* 24.1 (2024), e202400016. DOI: 10.1002/pamm.202400016.

[5] R. Beinert, J. Bresch, and G. Steidl. "Denoising of sphere- and SO(3)-valued data by relaxed Tikhonov regularization". In: *Inverse Probl. Imaging* 19.1 (2025), pp. 87–108. DOI: 10.3934/ipi.2024026.

[6] R. Bergmann, J. Persch, and G. Steidl. "A parallel Douglas-Rachford algorithm for minimizing ROF-like functionals on images with values in symmetric Hadamard manifolds". In: *SIAM J. Imaging Sci.* 9.3 (2016), pp. 901–937. DOI: 10.1137/15M1052858.

[7] R. Bergmann et al. "Second order differences of cyclic data and applications in variational denoising". In: *SIAM J. Imaging Sci.* 7.4 (2014), pp. 2916–2953. DOI: 10.1137/140969993.

[8] C. Carathéodory. "Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen". In: *Rendiconti del Circolo Matematico di Palermo (1884-1940)* 32.1 (1911), pp. 193–217. DOI: 10.1007/BF03014795.

[9] Rustum Choksi, Yves van Gennip, and Adam Oberman. *Anisotropic total variation regularized L^1-approximation and denoising/deblurring of 2d bar codes.* arXiv:1007.1035. 2017. DOI: 10.48550/arXiv.1007.1035.

[10] Laurent Condat. "A direct algorithm for 1D total variation denoising". 2012. URL: https://hal.science/hal-00675043.

[11] Laurent Condat. "A direct algorithm for 1D total variation denoising". In: *IEEE Signal Process. Lett.* 20.11 (2013), pp. 1054–1057. DOI: 10.1109/LSP.2013.2278339.

[12] Laurent Condat. "Tikhonov regularization of circle-valued signals". In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782. DOI: 10.1109/TSP.2022.3179816.

[13] Robin Kenis, Emanuel Laude, and Panagiotis Patrinos. *Convex relaxations for large-scale graphically structured nonconvex problems with spherical constraints: An optimal transport approach.* arXiv:2308.00079. 2023. DOI: 10.48550/arXiv.2308.00079.

[14] R. Meziat. "The method of moments in global optimization". In: *J. Math. Sci.* 116.3 (2003), pp. 3303–3324. DOI: 10.1023/A:1023621121247.