Denoising Hyperbolic-valued Data by Relaxed Regularizations

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GAMM 2025 95th Annual Meeting — Poznan University of Technology, Poland

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September 21, 2025

- **Focus:** important variational model with quadratic TIKHONOV and TOTAL VARIATION (TV) regularization.
- **Property of the solution:** values are close to each other ((un-)smoothly).
- **Disadvantage:** difficult nonconvex problem, due to the nonconvex manifold constraints, here, the hyperboloid.
- **Done 1):** nicely simplified relaxed model without loosing any information for TIKHONOV[1] for the sphere and SO(3).
- **Done 2):** improvement for the TV[2], w.r.t. geodesic models, e.g. for the $\mathbb{S}_0$—with "tightness" and applications for any sphere.
- **Aim 1):** application to more complicated problems, i.e., manifolds, here, e.g., the hyperboloid $\mathbb{H}_d := \{\mathbb{R}^{d+1} \mid \boldsymbol{x}_1^2 + \ldots + \boldsymbol{x}_d^2 = \boldsymbol{x}_{d+1}^2 - 1, \boldsymbol{x}_{d+1} > 0\}$.
- **Aim 2):** overview and motivation for a bunch of other manifold.

[1] Robert Beinert, Jonas Bresch, and Gabriele Steidl. "Denoising of sphere- and SO(3)-valued data by relaxed Tikhonov regularization". In: *Inverse Probl. Imaging* (2024). ISSN: 1930-8337. DOI: 10.3934/ipi.2024026.

[2] Robert Beinert and Jonas Bresch. "Denoising sphere-valued data by relaxed total variation regularization". In: *Proceedings in Applied Mathematics and Mechanics* 24.4 (2024). DOI: 10.1002/pamm.202400016.

- Let $G = (V, E)$ be a connected, undirected graph,
- with vertices $V := \{1, \ldots, N\}$, i.e. $N := |V|$,
- and edges $E \subset \{(n,m) \in V \times V : n < m\}$, define $M := |E|$.
- general denoising model, for instance G is the line- or grid-graph.

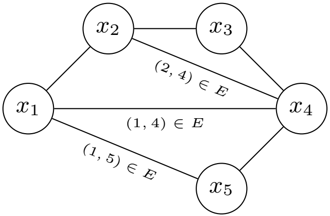

FIGURE 1: x_i manifold-valued, identified by $i \in V$.

- **Former approach by Condat**[3]**:** denoise a disturbed $\mathbb{S}_1$-valued signal on G, i.e. $y = (y_n)_{n \in V} \in \mathbb{C}^N$ using TIKHONOV and $\mathbb{S}_1 \hookrightarrow \mathbb{C}$.
- **First approach:** $\mathbb{S}_1$ can be embedded into $\mathbb{R}^2$, and model can be simplified [1].
- **Second approach:** theory can be generalized for sphere and SO(3)-valued data [1].
- **Third approach:** application for TV regularization [2][4].

[3] Laurent Condat. "Tikhonov regularization of circle-valued signals". In: *IEEE Trans. Signal Process.* 70 (2022), pp. 2775–2782. DOI: 10.1109/TSP.2022.3179816.

[4] Robert Beinert and Jonas Bresch. *Denoising hyperbolic-valued data by relaxed regularizations*. 2024. arXiv: 2410.16149 [math.NA]. URL: https://arxiv.org/abs/2410.16149.

Non-convex (denoising) problem over manifold

Step 1: use manifold constraints to **simplify/rewrite** the objective

Step 2: **extend** the set of feasible points to **linearize** the objective

Step 3: find enrich **matrix representation** for the set of feasible points

Step 4: **neclegt** non-convex constraints form the matrix-representation

convex-relaxed problem for denoising

- **Aim:** generalization via $\mathbb{R}^2 \hookleftarrow \mathbb{S}_1 := \{\boldsymbol{z} \in \mathbb{R}^2 : \|\boldsymbol{z}\| = 1\} \simeq \mathbb{S}_{\mathbb{C}}$.
- We aim to "solve" the following **nonconvex** denoising problem

$$\underset{\boldsymbol{x} \in \mathbb{S}_1^N}{\arg\min} \sum_{n \in V} \frac{w_n}{2} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \sum_{(n,m) \in E} \frac{\lambda_{(n,m)}}{2} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2. \tag{1}$$

- Realizing the complex multiplication using the matrix representation of $z \in \mathbb{C}$ [3]

$$\boldsymbol{M}(z) := \boldsymbol{M}(\boldsymbol{z}) := \begin{bmatrix} \Re[\boldsymbol{z}] & -\Im[\boldsymbol{z}] \\ \Im[\boldsymbol{z}] & \Re[\boldsymbol{z}] \end{bmatrix} \Rightarrow \boldsymbol{M}(\boldsymbol{z})\boldsymbol{x} = \overrightarrow{zx} = \begin{bmatrix} \Re[zx] \\ \Im[zx] \end{bmatrix}. \tag{2}$$

- **Introducing:** $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n \in \mathbb{R}^2$, yields the real of the complex-valued version:

$$(1) = \underset{\boldsymbol{x} \in \mathbb{S}_1^N, \boldsymbol{r} \in (\mathbb{R}^2)^M}{\arg\min} \mathcal{J}(\boldsymbol{x}, \boldsymbol{r}) \text{ s.t. } \boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n \quad \forall (n,m) \in E$$

with $\mathcal{J}(\boldsymbol{x}, \boldsymbol{r}) := -\sum_{n \in V} w_n \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle - \sum_{(n,m) \in E} \lambda_{(n,m)} \Re[\boldsymbol{r}_{(n,m)}]$.

Lemma 1 (Beinert, B., Steidl[1])

Let $n, m \in V$ *and* $(n,m) \in E$. *Then* $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_1$ *and* $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n$ *iff the block matrix*

$$\boldsymbol{P}_{(n,m)} := \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{x}_n) & \boldsymbol{M}(\boldsymbol{x}_m) \\ \boldsymbol{M}(\boldsymbol{x}_n)^{\mathrm{T}} & \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{r}_{(n,m)})^{\mathrm{T}} \\ \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} & \boldsymbol{M}(\boldsymbol{r}_{(n,m)}) & \boldsymbol{I}_2 \end{bmatrix} \in \mathbb{R}^{6 \times 6} \tag{3}$$

is positive semi-definite and has rank two.

Proof.

Using Schur's theorem to neglect the $\boldsymbol{M}(\cdot)$ and translate the positive semidefiniteness. The rank property, is, due to the $\boldsymbol{I}_2$, a "lowest"-rank property, such that the Schur complement is a full-zero block. □

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- **Note:** second components of $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n$ in the **relaxed real model** originate form the complex-valued model.
- **Claim:** equivalent real-valued problem

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{S}_1^N, \boldsymbol{\ell} \in \mathbb{R}^M} \mathcal{K}(\boldsymbol{x}, \boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m) \in E \tag{4}$$

  with $\mathcal{K}(\boldsymbol{x}, \boldsymbol{\ell}) \coloneqq -\sum_{n \in V} w_n \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle - \sum_{(n,m) \in E} \lambda_{(n,m)} \boldsymbol{\ell}_{(n,m)}$.
- **Encoding:** nonconvex constraints and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle$:

Lemma 2 (Beinert, B., Steidl[1])

Let $n, m \in V$ and $(n,m) \in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_1$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle$ iff

$$\boldsymbol{Q}_{(n,m)} \succeq 0 \textit{ and } \operatorname{rk}(\boldsymbol{Q}_{(n,m)}) = 2, \textit{ where } \boldsymbol{Q}_{(n,m)} \coloneqq \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & 1 & \boldsymbol{\ell}_{(n,m)} \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{\ell}_{(n,m)} & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4}. \tag{5}$$

- **Again:** neglect the rank-two constraint, we propose **our simplified relaxed real model**:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N, \boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E. \tag{6}$$

- Our convex model is simpler w.r.t. to the dimension of the matrix representation than

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N, \boldsymbol{r}\in(\mathbb{R}^2)^M} \mathcal{J}(\boldsymbol{x},\boldsymbol{r}) \quad \text{s.t.} \quad \boldsymbol{P}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E, \tag{7}$$

THEOREM 3 (BEINERT, B., STEIDL[1])

Both models are equivalent in the following sense:

(i) If $(\hat{\boldsymbol{x}},\hat{\boldsymbol{r}})$ *solves* (7), *then* $(\hat{\boldsymbol{x}},\Re[\hat{\boldsymbol{r}}])$ *solves* (6).

(ii) If $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{\ell}})$ *solves* (6), *then* $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{r}})$ *with* $\tilde{\boldsymbol{r}}_{(n,m)} = (\tilde{\boldsymbol{\ell}}_{(n,m)}, \Im[\boldsymbol{M}(\tilde{\boldsymbol{x}}_m)^{\mathrm{T}}\tilde{\boldsymbol{x}}_n])^{\mathrm{T}}$ *solves* (7).

COROLLARY 4

The ***simplified relaxed real model*** *and the* ***relaxed complex model*** *are equivalent.*

– **Main advantage:** the **simplified relaxed real model** is simple to generalize to $(d-1)$-dimensions; let $\mathbb{S}_{d-1} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| = 1\}$.

– The **original nonconvex problem** reads as

$$\begin{aligned} &\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_{d-1}^N} \sum_{n\in V} \frac{w_n}{2} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \sum_{(n,m)\in E} \frac{\lambda_{(n,m)}}{2} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2 \\ = &\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_{d-1}^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m) \in E \end{aligned}$$

Lemma 5 (Beinert, B., Steidl[2])

Let $n, m \in V$ and $(n,m) \in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_{d-1}$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle$ iff

$$\boldsymbol{Q}_{(n,m)} \coloneqq \begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & 1 & \boldsymbol{\ell}_{(n,m)} \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{\ell}_{(n,m)} & 1 \end{bmatrix} \in \mathbb{R}^{(d+2)\times(d+2)}$$

is positive semi-definite and has rank d.

- Let us stress to have a common embedding $\mathbb{M} \hookrightarrow \mathbb{R}^{d_{\mathbb{M}}}$ for some dimension $d_{\mathbb{M}} \in \mathbb{N}$
- We consider noisy data $\boldsymbol{y} = (\boldsymbol{y}_n)_{n\in\mathbb{N}}$ which might be not $\mathbb{M}$-valued. We aim to solve

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N} \frac{1}{2} \sum_{n\in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \frac{\lambda}{2} \sum_{(n,m)\in V} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2 \tag{8}$$

- Consider rewriting of (8) by using

$$\tfrac{1}{2}\|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 = \tfrac{1}{2}\boldsymbol{v}_n - \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle + const. \quad \text{if } \|\boldsymbol{x}_n\|^2 = \boldsymbol{v}_n.$$

- Utilizing the latter, yields

$$(8) = \operatorname*{arg\,min}_{\substack{\boldsymbol{x}\in\mathbb{M}^N,\\ \boldsymbol{v}\in\mathbb{R}^N,\\ \boldsymbol{\ell}\in\mathbb{R}^\bullet}} \underbrace{\sum_{n\in V} \left(\tfrac{\boldsymbol{v}_n}{2} - \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle\right) + \lambda \sum_{(n,m)\in V} \left(\tfrac{\boldsymbol{v}_n}{2} + \tfrac{\boldsymbol{v}_m}{2} - \boldsymbol{\ell}_{(n,m)}\right)}_{=:\mathcal{K}(\boldsymbol{x},\boldsymbol{v},\boldsymbol{\ell})} \quad \text{s.t.} \quad \begin{cases} \boldsymbol{v}_n = \|\boldsymbol{x}_n\|^2, \\ \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \end{cases}$$

- Novel matrix representations $\boldsymbol{Q}_{(n,m)}$ yield[5]

$$(8) = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{R}^{d\times N}, \boldsymbol{v}\in\mathbb{R}^N, \boldsymbol{\ell}\in\mathbb{R}^\bullet} \mathcal{K}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0, \text{min. } \operatorname{rk}(\boldsymbol{Q}_{(n,m)}) \quad \forall (n,m) \in E$$

[5] Robert Beinert and Jonas Bresch. *Denoising hyperbolic-valued data by relaxed regularizations.* 2024. arXiv: 2410.16149 [math.NA]. URL: https://arxiv.org/abs/2410.16149.

$\mathbb{M}$	Embedding	$\boldsymbol{V}_n, n \in V$	$\boldsymbol{Q}_{n,m}, (n,m) \in E$	Remark
$\mathbb{S}_{d-1}$	$\mathbb{R}^d$	$\begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n \\ \boldsymbol{x}_n^{\mathrm{T}} & 1 \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & 1 & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \\ \boldsymbol{x}_m^{\mathrm{T}} & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle & 1 \end{bmatrix}$	
$\mathbb{H}_d$	$\mathbb{R}^{d+1}$	$\begin{bmatrix} \boldsymbol{I}_{d+1} & \boldsymbol{x}_n & \tilde{\boldsymbol{x}}_n \\ \boldsymbol{x}_n^{\mathrm{T}} & \Vert \boldsymbol{x}_n \Vert^2 & -1 \\ \tilde{\boldsymbol{x}}_n^{\mathrm{T}} & -1 & \Vert \boldsymbol{x}_n \Vert^2 \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \tilde{\boldsymbol{x}}_n & \boldsymbol{x}_m & \tilde{\boldsymbol{x}}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & \Vert \boldsymbol{x}_n \Vert^2 & -1 & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle_{\mathbb{H}_d} \\ \tilde{\boldsymbol{x}}_n^{\mathrm{T}} & -1 & \Vert \boldsymbol{x}_n \Vert^2 & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle_{\mathbb{H}_d} & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \\ \boldsymbol{x}_m^{\mathrm{T}} & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle_{\mathbb{H}_d} & \Vert \boldsymbol{x}_m \Vert^2 & -1 \\ \tilde{\boldsymbol{x}}_m^{\mathrm{T}} & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle_{\mathbb{H}_d} & \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle & -1 & \Vert \boldsymbol{x}_m \Vert^2 \end{bmatrix}$	
$\mathbb{A}_d$	$\mathbb{R}^d$	$\begin{bmatrix} 1 & \boldsymbol{x}_n^{\mathrm{T}} \\ \boldsymbol{x}_n & \boldsymbol{I}_d + (x_{n,i} x_{n,j})_{i \neq j} \end{bmatrix}$	$\begin{bmatrix} 1 & \boldsymbol{x}_n^{\mathrm{T}} & \boldsymbol{x}_m^{\mathrm{T}} \\ \boldsymbol{x}_n & \boldsymbol{I}_d + (x_{n,i} x_{n,j})_{i \neq j} & \boldsymbol{x}_n \boldsymbol{x}_m^{\mathrm{T}} \\ \boldsymbol{x}_m & \boldsymbol{x}_n \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{I}_d + (x_{m,i} x_{m,j})_{i \neq j} \end{bmatrix}$	
$\mathbb{T}_d$	$\mathbb{R}^{2 \times d}$	$\begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{x}_n \\ \boldsymbol{x}_n^{\mathrm{T}} & \boldsymbol{I}_d + (\langle \boldsymbol{x}_{n,i}, \boldsymbol{x}_{n,j} \rangle)_{i \neq j} \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & \boldsymbol{I}_d + (\langle \boldsymbol{x}_{n,i}, \boldsymbol{x}_{n,j} \rangle)_{i \neq j} & \boldsymbol{x}_n^{\mathrm{T}} \boldsymbol{x}_m \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{x}_m^{\mathrm{T}} \boldsymbol{x}_n & \boldsymbol{I}_d + (\langle \boldsymbol{x}_{m,i}, \boldsymbol{x}_{m,j} \rangle)_{i \neq j} \end{bmatrix}$	$\mathbb{T}_d \simeq \times_{i=1}^d \mathbb{S}_1$
$\mathbb{V}_d(k)$	$\mathbb{R}^{d \times k}$	$\begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n \\ \boldsymbol{x}_n^{\mathrm{T}} & \boldsymbol{I}_k \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & \boldsymbol{I}_k & \boldsymbol{x}_n^{\mathrm{T}} \boldsymbol{x}_m \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{x}_m^{\mathrm{T}} \boldsymbol{x}_n & \boldsymbol{I}_k \end{bmatrix}$	$\mathbb{V}_d(1) = \mathbb{S}_{d-1}$
$\mathrm{SO}(3)$	$Q(8) \hookrightarrow \mathbb{S}_3$			other dimensions, too.
$\mathrm{SE}(2)$	$\mathbb{S}_1 \times \mathbb{R}^2$	$\begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{x}_n^{(12)} & \boldsymbol{x}_n^{(34)} \\ \boldsymbol{x}_n^{(12)\,\mathrm{T}} & 1 & \langle \boldsymbol{x}_n^{(12)}, \boldsymbol{x}_n^{(34)} \rangle \\ \boldsymbol{x}_n^{(34)\,\mathrm{T}} & \langle \boldsymbol{x}_n^{(12)}, \boldsymbol{x}_n^{(34)} \rangle & \Vert \boldsymbol{x}_n^{(34)} \Vert^2 \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & \boldsymbol{I}_k & \boldsymbol{x}_n^{\mathrm{T}} \boldsymbol{x}_m \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{x}_m^{\mathrm{T}} \boldsymbol{x}_n & \boldsymbol{I}_k \end{bmatrix}$	other dimensions, too.

Proposition 1

Let $\mathbb{M}$ be one of the considered manifolds, $d_{\mathbb{M}} \in \mathbb{N}$ the dimension of the used embedding in the Euclidean vector space and $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{R}^{d_{\mathbb{M}}}$. Then, $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{M}$ for $(n,m) \in E$ iff $\boldsymbol{Q}_{(n,m)}$ is positive semidefinite and has minimal rank.

- The TIKHONOV regularization (as in the latter section) is suitable for smooth signals.
- We want to **recover piecewise constant signals** from noisy measurements.
- For this reason, we **replace the squared 2-norm** of the regularizer **by the 1-norm**.
- Yielding an anisotropic TOTAL VARIATION (TV) regularization:

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{M}^N} \frac{1}{2} \sum_{n \in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|_2^2 + \mu \mathrm{TV}(\boldsymbol{x}) \quad \text{with} \quad \mathrm{TV}(\boldsymbol{x}) \coloneqq \sum_{(n,m) \in E} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|_1. \tag{9}$$

- Using the property of the squared 2-norm again, we obtain

$$\mathcal{K}(\boldsymbol{x}, \boldsymbol{v}) \coloneqq \sum_{n \in V} \left(\tfrac{\boldsymbol{v}_n}{2} - \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle\right) + \mu \mathrm{TV}(\boldsymbol{x}) \quad \text{with} \quad (9) = \operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{M}^N, \boldsymbol{v} \in \mathbb{R}^N} \mathcal{K}(\boldsymbol{x}, \boldsymbol{v}) \quad \text{s.t} \quad \boldsymbol{v}_n = \|\boldsymbol{x}_n\|^2.$$

PROPOSITION 2

Let $\mathbb{M}$ be one of the considered manifolds, $d_{\mathbb{M}} \in \mathbb{N}$ the dimension of the used embedding in the Euclidean vector space and $\boldsymbol{x}_n \in \mathbb{R}^{d_{\mathbb{M}}}$. Then, $\boldsymbol{x}_n \in \mathbb{M}$ for $n \in V$ iff $\boldsymbol{V}_n$ is positive semidefinite with minimal rank.

- Convexifying the $\mathbb{M}$-valued domain, we propose **our relaxed, convex problem**:

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{R}^{d_{\mathbb{M}} \times N}, \boldsymbol{v} \in \mathbb{R}^N} \mathcal{K}(\boldsymbol{x}, \boldsymbol{v}) \quad \text{s.t.} \quad \boldsymbol{V}_n \succeq 0 \quad \text{for all} \quad n \in V. \tag{10}$$

- Series of K noisy instances, assume each pixel $x_{i,j}^{(k)}$ follows a Gaussian distribution.
- Empirical mean and variance may be estimated

$$\hat{\mu}_{ij} := \frac{1}{K}\sum_{k=1}^{K} x_{i,j}^{(k)}, \qquad \hat{\sigma}_{ij}^2 := \frac{1}{K}\sum_{k=1}^{K}(x_{i,j}^{(k)} - \hat{\mu}_{ij})^2. \tag{11}$$

- Set of Gaussians $\mathcal{N}(\mu, \sigma)$ parameterized by $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_{>0}$ and equipped with the so-called Fisher metric is isometric to the hyperbolic space $\mathbb{H}_2$.
- The isometry is given by $\pi_3 \circ \pi_2 \circ \pi_1 : \mathcal{N} \to \mathbb{H}_2 := \{\boldsymbol{\xi} \in \mathbb{R}^3 : \boldsymbol{\xi}_1^2 + \boldsymbol{\xi}_2^2 - \boldsymbol{\xi}_3^2 = -1, \boldsymbol{\xi}_3 > 0\}$ with[6]

$$\pi_1 : \mathcal{N} \to \mathbb{P}_2, \qquad (\mu, \sigma) \mapsto \tfrac{1}{\sqrt{2}}(\mu, \sqrt{2}\sigma)^{\mathrm{T}}, \tag{12a}$$

$$\pi_2 : \mathbb{P}_2 \to \mathbb{D}_2, \qquad \boldsymbol{y} \mapsto \tfrac{1}{y_1^2 + (y_2+1)^2}(2y_1, \|\boldsymbol{y}\|_2^2 - 1)^{\mathrm{T}}, \tag{12b}$$

$$\pi_3 : \mathbb{D}_2 \to \mathbb{H}_2, \qquad \boldsymbol{y} \mapsto \tfrac{1}{1 - \|\boldsymbol{y}\|_2^2}(2y_1, 2y_2, 1 + \|\boldsymbol{y}\|_2^2)^{\mathrm{T}}. \tag{12c}$$

[6] Robert Beinert and Jonas Bresch. *Denoising hyperbolic-valued data by relaxed regularizations*. 2024. arXiv: 2410.16149 [math.NA]. URL: https://arxiv.org/abs/2410.16149.

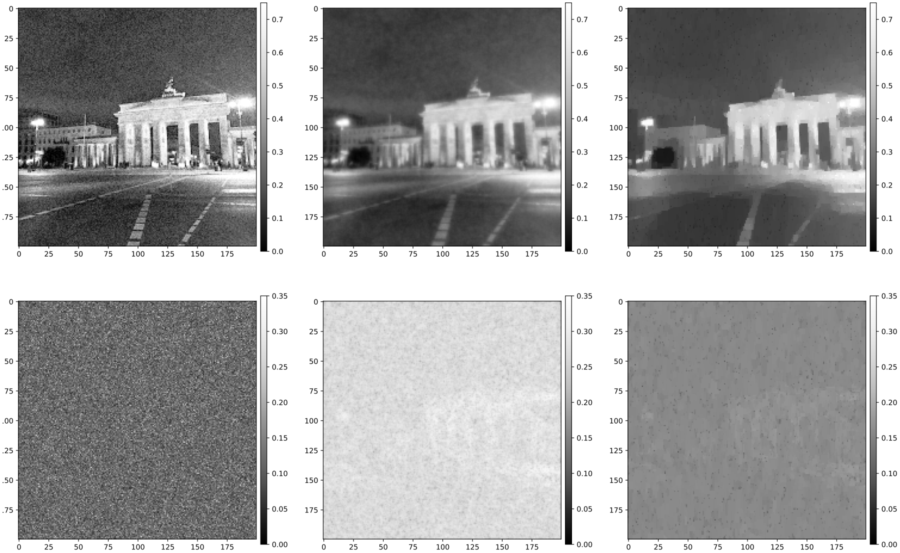

Figure 2: Empirical mean (top) and standard deviation (bottom) for the Brandenburg Gate image. Here, $K = 20$ shots are randomly sampled by adding white noise with $\sigma = 0.15$. The empirical quantities are shown on the left, the Tikhonov denoised images in the middle, and the TV denoised images on the right.

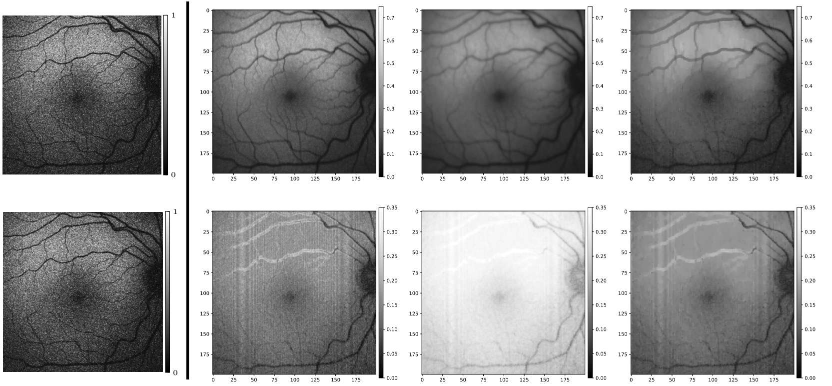

FIGURE 3: Clearing the estimated mean (top) and standard deviation (bottom) of $K = 20$ retina scans. *Leftmost:* the 1st (top) and 20th (bottom) image of the retina scans. Tikhonov (left) TV denoised (right).

TV-model						PDRA[7]			
SNR($\hat{\mu}$)	SNR($\hat{\sigma}$)	μ	$\mathcal{K}$	MAE $\eta \equiv -1$	time (sec.)	SNR($\hat{\mu}$)	SNR($\hat{\sigma}$)	α	time (sec.)
6.121	12.211	0.05	$2.61 \cdot 10^5$	$4.73 \cdot 10^{-4}$	160	5.963	12.400	0.1	120
6.001	12.369	0.10	$2.54 \cdot 10^5$	$4.78 \cdot 10^{-4}$	180	5.911	12.618	0.3	140
5.986	**12.301**	0.15	$1.81 \cdot 10^5$	$4.47 \cdot 10^{-4}$	190	5.874	12.748	0.5	280
5.966	12.455	0.20	$1.19 \cdot 10^5$	$4.67 \cdot 10^{-4}$	220	5.843	12.849	0.7	430

[7] R. Bergmann, J. Persch, and G. Steidl. "A parallel Douglas-Rachford algorithm for minimizing ROF-like functionals on images with values in symmetric Hadamard manifolds". In: *SIAM J. Imaging Sci.* 9.3 (2016), pp. 901–937. DOI: 10.1137/15M1052858.

Thank you for your attention!

[1] Robert Beinert and Jonas Bresch. *Denoising hyperbolic-valued data by relaxed regularizations.* 2024. arXiv: 2410.16149 [math.NA]. URL: https://arxiv.org/abs/2410.16149.

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[3] Robert Beinert, Jonas Bresch, and Gabriele Steidl. "Denoising of sphere- and SO(3)-valued data by relaxed Tikhonov regularization". In: *Inverse Probl. Imaging* (2024). ISSN: 1930-8337. DOI: 10.3934/ipi.2024026.

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