DENOISING HYPERBOLIC-VALUED DATA BY RELAXED REGULARIZATIONS

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Key Takeaways

- We introduce novel denoisers for data originally located on the hyperbolic sheet.
- Our models use similar techniques as the sphere- and SO(3)-valued data denoisers in [1, 2].
- The new denoiser is applied to real-world datasets in the context of Gaussian image processing.

Hyperbolic-Valued Data Processing

- The *Minkowski bilinear form* is defined as

$$\eta\colon \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}, \quad \eta(\boldsymbol{x}, \boldsymbol{y}) \coloneqq \sum_{i=1}^{d} x_i y_i - x_{d+1} y_{d+1},$$

- and the *hyperbolic sheet* via

$$\mathbb{H}_d \coloneqq \left\{\boldsymbol{x} \in \mathbb{R}^{d+1} \,\middle|\, \eta(\boldsymbol{x}, \boldsymbol{x}) = -1,\ x_{d+1} > 0\right\} \subset \mathbb{A}_{d+1} \coloneqq \mathbb{R}^d \times \mathbb{R}_{\geq 1}.$$

- We consider data on a connected, undirected graph (V, E) with

$$V \coloneqq \{1, \dots, N\}, \quad E \subseteq \{(n, m) \in V \times V : n < m\}, \quad |E| = M$$

- and aim to recover

$$\boldsymbol{x} \coloneqq (\boldsymbol{x}_n)_{n \in V} \in \mathbb{H}_d^N \quad \text{from noisy} \quad \boldsymbol{y} \coloneqq (\boldsymbol{y}_n)_{n \in V} \in (\mathbb{R}^{d+1})^N.$$

Tikhonov Denoiser

- For smooth data, we study the non-convex Tikhonov formulation

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{H}_d^N} \ \frac{1}{2} \sum_{n \in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|_2^2 + \frac{\lambda}{2} \sum_{(n,m) \in E} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|_2^2. \tag{1}$$

- Using auxiliary $\boldsymbol{f} \in \mathbb{R}^M$ and $\boldsymbol{v} \in \mathbb{R}^N$ as well as the linear functional

$$\mathcal{J}(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{v}) \coloneqq \frac{1}{2} \sum_{n \in V} \left(\boldsymbol{v}_n - 2\langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle\right) + \frac{\lambda}{2} \sum_{(n,m) \in E} \left(\boldsymbol{v}_n + \boldsymbol{v}_m - 2 \boldsymbol{f}_{(n,m)}\right),$$

we rewrite (1) as

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{H}_d^N} \ \mathcal{J}(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{v}) \ \text{ s.t. } \begin{Bmatrix} \boldsymbol{v}_n = \|\boldsymbol{x}_n\|_2^2 & \forall n \in V \\ \boldsymbol{f}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle & \forall (n, m) \in E \end{Bmatrix}.$$

- To encode the non-convex constraints, we introduce

$$\boldsymbol{Q}_{(n,m)} \coloneqq \left[\begin{array}{c|cc|cc} \boldsymbol{I}_{d+1} & \boldsymbol{x}_n & \tilde{\boldsymbol{x}}_n & \boldsymbol{x}_m & \tilde{\boldsymbol{x}}_m \\ \hline \boldsymbol{x}_n^* & \boldsymbol{v}_n & -1 & \boldsymbol{f}_{(n,m)} & \ell_{(n,m)} \\ \tilde{\boldsymbol{x}}_n^* & -1 & \boldsymbol{v}_n & \ell_{(n,m)} & \boldsymbol{f}_{(n,m)} \\ \hline \boldsymbol{x}_m^* & \boldsymbol{f}_{(n,m)} & \ell_{(n,m)} & \boldsymbol{v}_m & -1 \\ \tilde{\boldsymbol{x}}_m^* & \ell_{(n,m)} & \boldsymbol{f}_{(n,m)} & -1 & \boldsymbol{v}_m \end{array}\right] \in \mathbb{R}^{(d+5)\times(d+5)},$$

where $\tilde{\boldsymbol{\xi}} \coloneqq (\xi_1, \dots, \xi_d, -\xi_{d+1})^*$.

Proposition (Non-convex constraints via $\boldsymbol{Q}_{(n,m)}$)

Let $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{A}_{d+1}$. Then

$$\begin{Bmatrix} \boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{H}_d,\ \boldsymbol{v}_n = \|\boldsymbol{x}_n\|_2^2,\ \boldsymbol{v}_m = \|\boldsymbol{x}_m\|_2^2 \\ \ell_{(n,m)} = \eta(\boldsymbol{x}_n, \boldsymbol{x}_m),\ \boldsymbol{f}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \end{Bmatrix} \iff \begin{Bmatrix} \boldsymbol{Q}_{(n,m)} \succeq \boldsymbol{0} \\ \operatorname{rk}(\boldsymbol{Q}_{(n,m)}) = d + 1 \end{Bmatrix}.$$

- Neglecting the rank, we propose the relaxed convex model:

$$\operatorname*{arg\,min}_{\substack{\boldsymbol{x} \in \mathbb{A}_{d+1}^N,\ \ell \in \mathbb{R}^M, \\ \boldsymbol{f} \in \mathbb{R}^M,\ \boldsymbol{v} \in \mathbb{R}^N}} \ \mathcal{J}(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{v}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq \boldsymbol{0} \quad \forall (n, m) \in E.$$

Total Variation Denoiser

- For cartoon-like data, we study the total variation formulation

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{H}_d^N} \frac{1}{2} \sum_{n \in V} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|_2^2 + \mu \operatorname{TV}(\boldsymbol{x}), \quad \operatorname{TV}(\boldsymbol{x}) \coloneqq \sum_{(n,m) \in E} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|_1. \tag{2}$$

- Using auxiliary $\boldsymbol{v} \in \mathbb{R}^N$ and the convex functional

$$\mathcal{K}(\boldsymbol{x}, \boldsymbol{v}) \coloneqq \frac{1}{2} \sum_{n \in V} \left(\boldsymbol{v}_n - 2\langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle\right) + \mu \operatorname{TV}(\boldsymbol{x}),$$

we rewrite (2) as

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{H}_d^N} \ \mathcal{K}(\boldsymbol{x}, \boldsymbol{v}) \quad \text{s.t.} \quad \boldsymbol{v}_n = \|\boldsymbol{x}_n\|_2^2 \quad \forall n \in V.$$

- To encode the non-convex constraints, we introduce

$$\boldsymbol{V}_n \coloneqq \left[\begin{array}{c|cc} \boldsymbol{I}_{d+1} & \boldsymbol{x}_n & \tilde{\boldsymbol{x}}_n \\ \hline \boldsymbol{x}_n^* & \boldsymbol{v}_n & -1 \\ \tilde{\boldsymbol{x}}_n^* & -1 & \boldsymbol{v}_n \end{array}\right] \in \mathbb{R}^{(d+3)\times(d+3)}$$

Proposition (Non-convex constraints via $\mathrm{V_n}$)

Let $\boldsymbol{x}_n \in \mathbb{A}_{d+1}$. Then

$$\left\{\boldsymbol{x}_n \in \mathbb{H}_d,\ \boldsymbol{v}_n = \|\boldsymbol{x}_n\|_2^2\right\} \iff \left\{\boldsymbol{V}_n \succeq 0,\ \operatorname{rk}(\boldsymbol{V}_n) = d + 1\right\}.$$

- Neglecting the rank, we propose the relaxed convex model:

$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{A}_{d+1}^N,\ \boldsymbol{v} \in \mathbb{R}^N} \ \mathcal{K}(\boldsymbol{x}, \boldsymbol{v}) \quad \text{s.t.} \quad \boldsymbol{V}_n \succeq 0 \quad \forall n \in V.$$

Numerical Experiments

- We solve both convex relaxed formulations via ADMM.

Synthetic Data

- Smooth $\mathbb{H}_2$ line signal disturbed by Gaussian noise.

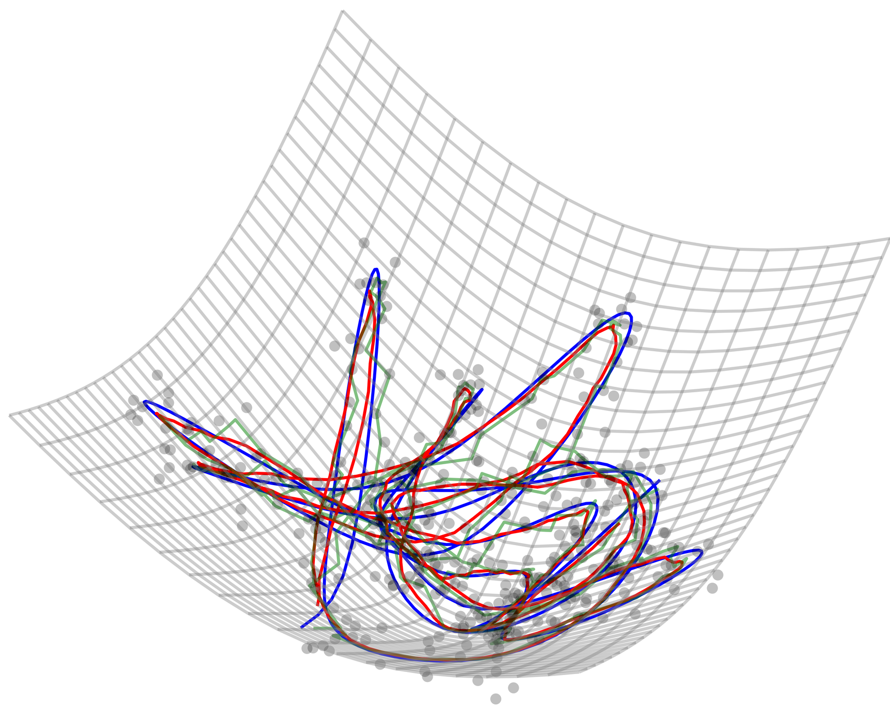

Figure: Restoration of true synthetic line signal (blue) from noisy measurements (gray dots); solution of the Tikhonov model (red), and the TV model (green).

Gaussian Image Processing

- We measure a series of K noisy images.
- Each pixel $x_{i,j}^{(k)}$ follows a Gaussian distribution with

$$\hat{\mu}_{ij} \approx \frac{1}{K} \sum_{k=1}^{K} x_{i,j}^{(k)}, \qquad \hat{\sigma}_{ij}^2 \approx \frac{1}{K} \sum_{k=1}^{K} \left(x_{i,j}^{(k)} - \hat{\mu}_{ij}\right)^2.$$

- The set of Gaussians $\mathcal{N}(\mu, \sigma)$ is isometric to $\mathbb{H}_2$.

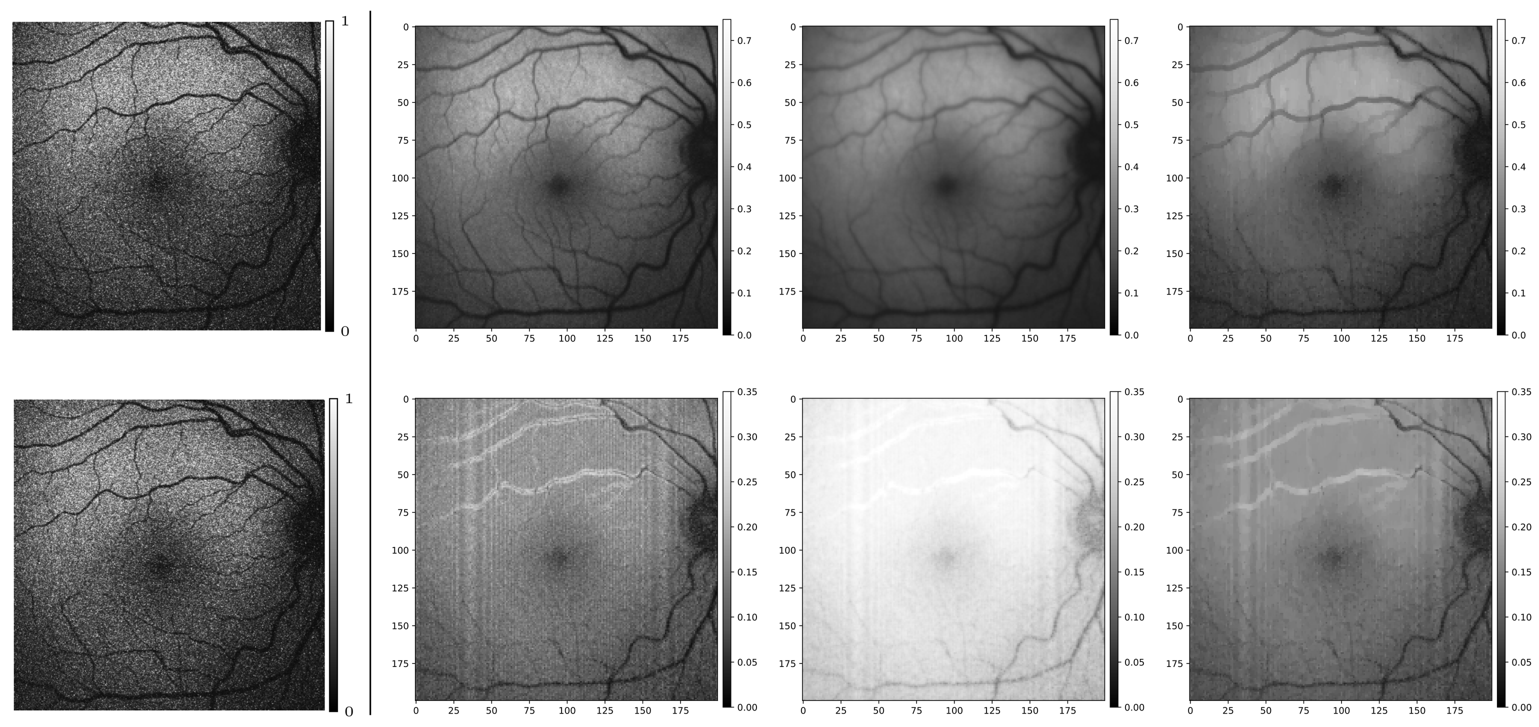

Figure: Denoising a series of $K = 20$ retina scans. *Leftmost:* the 1st (top) and 20th (bottom) scan. *From left to right:* the empirical mean (top) and standard deviation (bottom), Tikhonov denoised images, and TV denoised images.

Table: Comparison of the TV denoising and PDRA (Douglas–Rachford) [3] regarding the SNR for the restored mean and standard deviation. The marked instance ($\mu = 0.15$) is visualized above.

ADMM ($\rho = 1$)						PDRA ($\eta = 0.5, \lambda = 0.9$)			
SNR($\hat{\mu}$)	SNR($\hat{\sigma}$)	μ	$\mathcal{K}$	MAE $\eta \equiv -1$	time (sec.)	SNR($\hat{\mu}$)	SNR($\hat{\sigma}$)	α	time (sec.)
6.121	12.211	0.05	$2.61 \cdot 10^5$	$4.73 \cdot 10^{-4}$	160	5.963	12.400	0.1	120
6.001	12.369	0.10	$2.54 \cdot 10^5$	$4.78 \cdot 10^{-4}$	180	5.911	12.618	0.3	140
5.986	**12.301**	0.15	$1.81 \cdot 10^5$	$4.47 \cdot 10^{-4}$	190	5.874	12.748	0.5	280
5.966	12.455	0.20	$1.19 \cdot 10^5$	$4.67 \cdot 10^{-4}$	220	5.843	12.849	0.7	430

References

[1] R. Beinert and J. Bresch: Denoising sphere-valued data by relaxed total variation regularization. *Proc. Appl. Math. Mech.* (2024).

[2] R. Beinert, J. Bresch, G. Steidl: Denoising of sphere- and SO(3)-valued data by relaxed Tikhonov regularization. *Inverse Probl. Imaging* (2024).

[3] R. Bergmann, J. Persch, and G. Steidl: A parallel Douglas-Rachford algorithm for minimizing ROF-like functionals on images with values in symmetric Hadamard manifolds. *SIAM J. Imaging Sci.* (2016).

[4] R. Beinert and J. Bresch: Denoising hyperbolic-valued data by relaxed regularizations. *SSVM.* (2025) – arXiv:2410.16149.

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