Denoising Manifold-Valued Data by Relaxed Regularizations

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Reconstruction and denoising manifold-valued data is a common task in many applications. Different tasks require different data representations encoded naturally in several manifolds.

Phase and orientation measurements, *chromaticity* components of color images, and statistical pixel descriptors are commonplace areas for manifold-valued data. Circle-valued data arise in *interferometric synthetic aperture radar* (ISAR) in color restoration models based on the *hue-saturation-value* (HSV) or *lightness-chromaticity-hue* (LCh) color spaces, and in phase imaging for *magnetic resonance*. 3d-sphere-valued signals model directional data arising in 3d orientation and directional sensing problems and appear in *chromaticity-brightness* image models. The *special orthogonal group* (SO) encodes rotations and arises in motion analysis and *electron back-scatter diffraction* (EBSD) imaging while quaternion representations are sometimes used for color and orientation processing. In another direction, the Fisher-metric geometry of Gaußian distributions identifies a two-dimensional hyperbolic sheet as a natural manifold for pixelwise mean–variance models, connecting probabilistic image models to hyperbolic-geometry-based processing. Generalizing the $(d-1)$-dimensional sphere yields the Stiefel-manifold, which arises in many applications, for instance, in image and video-based recognition and for handling orthonormal constraints. Beyond "classical" manifold-valued data, multi-binary valued data, as generalization of binary-valued data, can be found in modern and secure information transmission systems, for instance in multi-colour *quick-response* (QR) codes. Instead of black and white QR codes—two-dimensional matrix barcodes (invented in 1994 by Hara)—the color channels in the RGB values (red-blue-green) are added to the color model yielding modules in black, red, blue, green, cyan, magenta, yellow, and white.

All these quite different applications motivate to formulate a unifite pathway for all the different underlying manifolds. The aim is to be as manifold- and geometry-aware as necessary but, on the other side, as relaxed as possible allowing the application of state-of-the-art solver from convex analysis with existing convergence guarantees.

For doing so, novel Gram matrix representations $\boldsymbol{Q}$ for the manifold-valued data are proposed via a *positive semidefinite* lifting including minimal rank constraints. Those are the nonconvex constraints from the manifold originating due to the Euclidean embedding of the considered manifolds ensured by Whitney's (strong) em-

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bedding theorem for any smooth real m-dimensional manifold.

There are two different regularizer considered: the squared Tikhonov and total variation (TV) regularization. Therefore, the following nonconvex optimization problem is aimed to solve:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{M}^N} \sum_{n\in V} \tfrac{w_n}{2}\|\boldsymbol{x}_n - \boldsymbol{y}_n\|_2^2 + \sum_{(n,m)\in E} \lambda_{(n,m)} \mathcal{D}(\boldsymbol{x}_n, \boldsymbol{x}_m), \tag{1}$$

where $\mathbb{M} \hookrightarrow \mathbb{R}^{d_\mathbb{M}}$ is the ambient Euclidean embedding with underlying data structure, given by a connected, undirected graph $G = (V, E)$ with $|V| \coloneqq N$ and $|E| \coloneqq M$. The chosen penalizers $\mathcal{D} : \mathbb{R}^{d_\mathbb{M}} \times \mathbb{R}^{d_\mathbb{M}} \to \mathbb{R}$ are

$$\mathcal{D}(\cdot^1, \cdot^2) \in \{\| \cdot^1 - \cdot^2 \|_1, 1/2\| \cdot^1 - \cdot^2 \|_2^2\},$$

i.e. the Manhattan and the (squared) Euclidean norm. Capturing both regularizers enables to handle smooth and non-smooth, i.e., e.g. piecewise constant, data. It holds [1–4]

$$(1) = \operatorname*{arg\,min}_{\substack{\boldsymbol{x}\in\mathbb{R}^{d_\mathbb{M}}\\ \boldsymbol{v}\in\mathbb{R}^N, \boldsymbol{\ell}\in\mathbb{R}^M}} \sum_{n\in V} \tfrac{w_n}{2}(\boldsymbol{v}_n - 2\langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle) + \sum_{(n,m)\in E} \lambda_{(n,m)} \mathcal{D}(\boldsymbol{x}_n, \boldsymbol{x}_m) =: \mathcal{K}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\ell}),$$

$$\text{s.t.} \quad \begin{cases} \boldsymbol{Q}_{(n,m)}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\ell}) \succcurlyeq \boldsymbol{0}, \\ \min \operatorname{rk}(\boldsymbol{Q}_{(n,m)}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\ell})), \end{cases} \quad \forall (n, m) \in E. \tag{2}$$

Here, $\boldsymbol{v}_n$ parametrizes $\|\boldsymbol{x}_n\|^2$ and $\boldsymbol{\ell}_{(n,m)}$ parametrizes $\langle \boldsymbol{x}_n, \boldsymbol{x}_m\rangle$. It is proposed [1–4] to neglect the rank constraints in (2) and to solve instead the

convex relaxed model

$$\operatorname*{arg\,min}_{\substack{\boldsymbol{x}\in\mathbb{R}^{d_\mathbb{M}}\\ \boldsymbol{v}\in\mathbb{R}^N, \boldsymbol{\ell}\in\mathbb{R}^M}} \mathcal{K}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\ell}) \succcurlyeq \boldsymbol{0} \quad \forall (n, m) \in E. \tag{3}$$

If the squared Euclidean norm is used, the function $\mathcal{K}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\ell})$ becomes linear in all components [1; 2; 4], due to

$$\|\boldsymbol{x}_n - \boldsymbol{x}_m\|_2^2 = \boldsymbol{v}_n + \boldsymbol{x}_m - 2\boldsymbol{\ell}_{(n,m)}.$$

Beyond theoretical *tightness results* for $\mathbb{M} = \mathbb{S}_0^d$ [2; 3], i.e. multi-ninary-valued data, meaning that solutions of (3) can be used to obtain global solutions of the original nonconvex optimization problem (1), the computational examples [1–4] admit across all experiments such a behavior and, more precisely, the computed solution are already manifold-valued.

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References

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